Universidad de Concepción
Dirección de Postgrado
Facultad de Ciencias Físicas y Matemáticas - Programa de Magister en Matemática

Topologías Estrictas en un Espacio de Funciones y una Representación Integral para Operadores Débilmente Compactos

(Strict Topologies on a Function Space and an Integral Representation of Weakly Compact Operators)

ANGEL DANIEL BARRÍA COMICHEO
CONCEPCIÓN-CHILE
2011

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Angel Daniel Barría Comicheo

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A mi familia.

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Introduction.

In 1958, R.C Buck ([5]) introduced the notion of strict topology on the space of all real-valued, continuous and bounded functions, with domain in a locally compact space. Between 1967 and 1972, different generalizations of this topology, given by D.H. Fremlin, A.C.M. Van Rooij and F.D. Sentilles ([9, 27, 24]), were defined on the space $C_b(X)$ of all real-valued, continuous and bounded functions, with domain on a completely regular space X. In the literature, these topologies are denoted by β_0, β and β_1.Sentilles achieved an identification of the dual $(C_b(X),\beta_0)'$, $(C_b(X),\beta)'$ and $(C_b(X),\beta_1)'$ with the measure spaces studied by V.S. Varadarajan ([28]) $M_t(X)$, $M_\tau(X)$ and $M_\sigma(X)$ respectively, through an integral representation of the functionals in the sense of Riesz .

In 1976, A. Katsaras ([15]) generalizes these topologies to the space $C_{rc}(X,E)$ of all continuous functions with relatively compact range and domain in a completely regular space X to values in a Hausdorff locally convex space E. The generalization of the topology β_0 in this space was denoted by $\beta_{\mathscr{F}}$. Katsaras proved that the bounded sets with respect to the topology $\beta_{\mathscr{F}}$ coincide with the uniformly bounded sets and, also, provided an integral representation for any functional belonging to $(C_{rc}(X,E),\beta_{\mathscr{F}})'$ with respect to a vector measure space.

Then, in 1986, this topology was generalized by J. Zafarani ([31]) to the space $C_b(X,E)$ of the bounded and continuous functions with domain in a completely regular space X and values in a Hausdorff locally convex space E. This new topology was denoted by $\beta_{\mathcal{P}}$.

The purpose of this thesis is to develop and complement the study of the space $(C_b(X,E),\beta_{\mathcal{P}})$ presented in [31] and study the representation of its dual as a measure space based on [15] and [8]. Also, it is planned to contribute in this theory showing an integral representation of weakly compact operators on the space $(C_b(X,E),\beta_{\mathcal{P}})$.

This document is structured as follows: Chapter 1 provides definitions and notations of the function spaces and the topologies that will be studied. It also presents the basic concepts of Baire sets and Baire measures.

Chapter 2 presents the necessary and sufficient conditions so the various topologies defined in $C_b(X,E)$ coincide. It is shown that a subset $C_b(X,E)$ is $\beta_{\mathcal{P}}$-bounded if and only if is τ_u-bounded. It is able to identify the spaces E and $(C_b(X),\gamma_{\mathcal{P}})$ as $\beta_{\mathcal{P}}$-closed subpaces of $C_b(X,E)$ and it is proved the $\beta_{\mathcal{P}}$-density of $C_b(X)\otimes E$. It is shown the necessary and sufficient conditions so the space $(C_b(X,E),\beta_{\mathcal{P}})$ is barreled, quasibarreled, DF-space, gDF-space, bornological among others. There are studied the necessary and sufficient conditions so the space $(C_b(X,E),\beta_{\mathcal{P}})$ is separable, complete, quasicomplete and sequentially complete. Also, it is obtained a characterization of relatively $\beta_{\mathcal{P}}$-compact sets. The chapter is ended showing the different characterizations of the topology β_p through various bases of neighborhoods of zero.

Chapter 3 introduces the concepts of $\mathcal{P}_p$-tight functional, $\mathcal{P}_p$-tight measure and $\mathcal{P}_p^q$-tight operator. It is identified the $\beta_{\mathcal{P}}$-continuity of linear operators and functionals on $C_b(X,E)$ with the conditions $\mathcal{P}_p^q$-tight and $\mathcal{P}_p$-tight respectively. Then, it is shown an equivalent definition of the integral for bounded functions of X in $\mathbb{R}$, respect to Baire measures, using convergent nets in the sense of Riemann sums. This concept of integration is generalized to functions of the space $C_b(X,E)$, respect to $\mathcal{P}_p$-tight vector measures. It is shown an integral representation of linear $\beta_{\mathcal{P}}$-continuous functionals respect to the $\mathcal{P}_p$-tight measures and are characterized the $\beta_{\mathcal{P}}$-equicontinuous sets. Also, it is shown that if F is a Frechet space, then, an operator of the type $T: C_b(X)\to F$ is weakly $(\tau_{\|\cdot\|},\tau_F)$-compact and $(\gamma_{\mathcal{P}},\tau_F)$-continuous if and only if is weakly $(\gamma_{\mathcal{P}},\tau_F)$-compact.

In the final part of the chapter, it is define a space of vector measures of the type $m:\mathcal{B}\to\mathscr{L}(E,F)$ and the concept of integral is extended for functions in $C_b(X,E)$ respect to these measures. Finally, it is given an integral representation in the sense of Riesz to weakly $(\beta_{\mathcal{P}}, \tau_{_F})$-compact $(\beta_{\mathcal{P}}, \tau_{_F})$-continuous operators defined in $C_b(X, E)$ with values in a Hausdorff locally convex space F, respect to certain vector measures and it is proved that this representation is unique.

Obtained results.

In Chapter 2, it is defined Γ as the family of all the directed coverings of X formed by compact sets and it is defined the relation $\preceq$ as follows: for $\mathcal{P}, \mathcal{P}' \in \Gamma$, $\mathcal{P} \preceq \mathcal{P}'$ if and only if $\mathcal{P}'$ is a refinement of $\mathcal{P}$. We obtain the following proposition:

Proposition 2.6.12

(a) If X is a $K_{\mathcal{P}}$-space and E is sequentially complete, then for each $\mathcal{P}_1 \in \Gamma$, such that $\mathcal{P}_1 \preceq \mathcal{P}$, the space $(C_b(X,E), \beta_{\mathcal{P}_1})$ is sequentially complete.

(b) If $\mathcal{P}$ have countable cofinal subfamily and $(C_b(X,E), \beta_{\mathcal{P}})$ is sequencially complete, then the space X is a $K_{\mathcal{P}}$-space and E is sequentially complete.

In [31] J. Zafarani proved the following result:

Proposition 2.6.11 *If $(C_b(X,E), \delta)$ is sequentially complete, then for each $\mathcal{P} \in \Gamma$, the space $(C_b(X,E), \beta_{\mathcal{P}})$ is sequentially complete.*

Using the result 2.6.12, we obtain the following remark:

Remark 2.6.13 *In general, the fact that there exists $\mathcal{P} \in \Gamma$, such that $(C_b(X,E), \beta_{\mathcal{P}})$ is sequentially complete, it is not implied that $(C_b(X,E), \delta)$ is sequentially complete.*

In Chapter 3 we will give a generalization of the Theorem 3.2 presented in [8, page 846] by Robert Fontenot, which only refers to the characterization of functionals on $(C_b(X,E), \beta_0)$ when E is normed. The obtained result is the following:

Theorem 3.1.3 *Let $T : C_b(X,E) \to F$ be a linear operator. Consider the following propositions:*

1. *T is $(\beta_{\mathcal{P}}, \tau_F)$-continuous.*

2. *For all $q \in s(F)$, there exists $p \in s(E)$ such that T is (β_p, q)-continuous.*

3. *For all $q \in s(F)$, there exists $p \in s(E)$ such that T is $\mathcal{P}_p^q$-tight.*

4. *Given $\varepsilon > 0$ and $q \in s(F)$ there exist $K \in \mathcal{P}$ and $p \in s(E)$ such that $q(T(f)) < \varepsilon$ for each $f \in C_b(X,E)$ which satisfies $\|f\|_p \leqslant 1$ and $f|_K = 0$.*

5. *Any net $(f_\alpha)_{\alpha\in\Lambda}$ in $\{f \in C_b(X) : \|f\| \leqslant 1\}$, such that is $\tau_{\mathcal{P}}$-convergent to 0, satisfy the following condition: for each $q \in s(F)$, there exists $p \in s(E)$, such that $(T(f_\alpha g))_{\alpha\in\Lambda}$ is q-convergent to 0 uniformly for $g \in \{f \in C_b(X,E) : \|f\|_p \leqslant 1\}$.*

6. *Assuming that $\mathbb{K} = F = \mathbb{R}$, the map $M : C_b(X) \to \mathbb{R}$ defined by $M(f) = \sup\{|T(g)| : g \in C_b(X,E)$ such that $\forall p \in s(E), \forall x \in X, p(g(x)) \leqslant f(x)\}$ to $f \geqslant 0$ and defined by $M(f) := M(f^+) - M(f^-)$ for $f \in C_b(X)$, is a linear functional $\mathcal{P}_p$-tight.*

Are satisfied the following statements:

(a) *The propositions $1, 2$ and 3 are equivalents.*

(b) *The proposition 3 implies the proposition 4.*

(c) *By assuming that $\mathbb{K} = \mathbb{R}$ and T is (τ_u, τ_F)-continuous, we have: $4 \Rightarrow 5 \Rightarrow 6$.*

(d) *By assuming that $\mathbb{K} = \mathbb{R}$ and E is normed with norm p, we have: $6 \Rightarrow 1$.*

It is important to note that if $\mathcal{P}$ is the collection of all compact subsets of X, $\mathbb{K} = \mathbb{R} = F$, T is an uniformly continuous functional and E is a normed space, then, the previous theorem is reduced to the result shown by Fontenot.

In Section 3.5 is considered a Frechet space F, whose topology is denoted by τ_F, and presents the following definition.

Definition 3.5.1 *Let τ be a vectorial topology on $C_b(X)$. An operator $T : C_b(X) \to F$ is (τ, τ_F)-weakly compact if there exists a τ-neighborhood V of zero such that $T(V)$ is relatively $\sigma(F, F')$-compact. We will simply say that T is weakly compact if it is $(\tau_{\|\cdot\|}, \tau_F)$-weakly compact.*

Proposition 3.5.2 *For a linear operator $T : C_b(X) \to E$ the following statements are equivalents:*

(a) T is weakly compact and $(\gamma_{\mathcal{P}}, \tau_F)$-continuous.

(b) T is $(\gamma_{\mathcal{P}}, \tau_F)$-weakly compact.

Additionally, we obtain

Proposition 3.5.3 *For a linear operator $T : C_b(X) \to E$ the following statements are equivalents:*

(a) T is compact and $(\gamma_{\mathcal{P}}, \tau_F)$-continuous.

(b) T is $(\gamma_{\mathcal{P}}, \tau_F)$-compact.

In Section 3.6 we present the following definition of weakly compact operator, which is a generalization of the definition 3.5.1.

Definition 3.6.7 *Let A and B be topological vector spaces and let $Bd(A)$ be the family of all the τ_A-bounded subsets of A. We will say that a linear operator $T : A \to B$ is* ***(τ_A, τ_B)-weakly compact****, if $T(S)$ is relatively $\sigma(B, B')$-compact, for each $S \in Bd(A)$.*

In this Section, we denote by $\mathscr{L}_{\beta_{\mathcal{P}},w}(C_b(X,E),F)$ the space of all weakly $(\beta_{\mathcal{P}}, \tau_F)$-compact and $(\beta_{\mathcal{P}}, \tau_F)$-continuous operators and the space of vector measures

$\mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E,F))$ is presented. The main result of the thesis is the following:

Theorem 3.6.12 *The map* $\Phi : \mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E,F)) \to \mathscr{L}_{\beta_{\mathcal{P}},w}(C_b(X,E), F)$*, defined by* $\Phi(m)(f) = \int_X f\, dm$*, is bijective and is such that:* $\forall x' \in F', x'm = \Phi(m)'x'$*. Also, if* $p \in s(E)$ *and* $q \in s(F)$ *are such that* $m_p^q(X) < \infty$*, then* $m_p^q(X) = \|\Phi(m)\|_p^q$*.*

Chapter 1

Notations and definitions.

1.1. Function Spaces.

Throughout the thesis, we assume knowledge of the basic tools of general topology, vector topologies and, more specifically, the locally convex spaces. Through this document, X will be denoted as a Hausdorff completely regular space, βX its Stone-$\check{C}$ech compactification, $\mathbb{K}$ the field of real numbers or complex numbers and E a Hausdorff non-trivial locally convex space on $\mathbb{K}$, which topology will be denoted by $\tau_{_E}$. The identity map in E will be denoted by Id_E and the topological dual of E will be denoted by E'. The Mackey topology in E respect to the duality $\langle E, E' \rangle$ will be denoted by $\tau(E, E')$. Additionally, the space $(E', \beta(E', E))$ will be denoted by E'_b. If $(H, \tau_{_H})$ and $(F, \tau_{_F})$ are Hausdorff locally convex spaces, then, the space of all the $(\tau_{_H}, \tau_{_F})$-continuous linear operators $T : H \to F$ will be denoted by $\mathscr{L}_{\tau_H}^{\tau_F}(H, F)$ and to mention that $V \subset H$ is a neighborhood of 0 respect to the topology τ_H, we will say that V is a τ_H-neighborhood of 0.

Given two topologies τ_1 and τ_2 defined in the same set, to say that τ_1 is coarser than τ_2 we will simply write $\tau_1 \leqslant \tau_2$. If K is a subset of X, then we will denote by $\mathscr{X}_K$ the characteristic function of K. For a family $\mathscr{R}$ of seminorms in a vector space V, it will be denoted by $\sigma(V, \mathscr{R})$ the coarser topology among all vector topologies that make continuous each seminorm of $\mathscr{R}$.

That is to say, $\sigma(V, \mathscr{R})$ is the topology which each $x \in V$ has a base of neighborhoods consisting of all sets of the form:

$$x + \{s \in V : p_i(s) \leqslant \varepsilon, \ \ i = 1...n\}$$

where $\{p_1, ..., p_n\} \in \mathscr{R}$ and $\varepsilon > 0$.

Definition 1.1.1. *Let $\mathscr{R}$ be a family of seminorms in a vector space V. We will say that $\mathscr{R}$ is* ***directed*** *(or* ***filtrant****) if for each pair of seminorms p_1 and p_2 in $\mathscr{R}$ there exists a seminorm p in $\mathscr{R}$ such that $p_1, p_2 \leqslant p$.*

The following proposition gives a useful characterization of the topologies that will be define in this document; however, we will omit its proof, since it is enought to apply the previous definition.

Proposition 1.1.2. *If $\mathscr{R}$ is a family of seminorms directed in the vector space V, then, the collection of all sets of the form*

$$\{z \in V : p(z) \leqslant \varepsilon\}$$

where $p \in \mathscr{R}$ and ε>0, is a base of neighborhoods of zero for the topology $\sigma(V, \mathscr{R})$.

Definition 1.1.3. *It will be said that a family $\mathscr{R}$ of seminorms in E is* ***generater*** *if $\sigma(E, \mathscr{R}) = \tau_{E}$.*

A classic example of a generater family in E is the family of all the continuous seminorms in E.

From now on $s(E)$ will be a family of continuous seminorms in E, generator, directed, that does not contain the null map. The space of bounded functions with domain X and values in E will be denoted by $B(X, E)$. In other words

$$B(X, E) := \left\{ f \in E^X : \forall p \in s(E), \ \sup_{x \in X} \{p(f(x))\} < \infty \right\}.$$

The space of bounded and continuous functions with domain X and values in E will be denoted by $C_b(X, E)$. In other words

$$C_b(X, E) := \{f \in B(X, E) : f \text{ is continuous }\}.$$

If E is a normed space, then $s(E)$ will have an only element which is the norm. In case that $E = \mathbb{K}$, we will write $B(X)$ to denote $B(X,\mathbb{K})$ and $C_b(X)$ to denote $C_b(X,\mathbb{K})$, understanding by $s(\mathbb{K}) = \{|\cdot|\}$ the set which only element is the absolute value or the complex module, as appropriate.

Definition 1.1.4. *For each function $f \in C_b(X)$, it is define the* ***support of f*** *as the set* $\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}$.

Consider $e \in E$ and $f \in C_b(X)$. Denoting by $f \otimes e$ the function defined by $f \otimes e : X \longrightarrow E$, $f \otimes e(x) = f(x)e$. Note that $f \otimes e \in C_b(X,E)$ since the composition of continuous and bounded functions.

1.2. Topologies in the spaces $B(X,E)$ and $C_b(X,E)$.

Definition 1.2.1. *Let p be a seminorm of $s(E)$. In $B(X,E)$ it is define the seminorm $\|\cdot\|_p$ as follows:*

$$\|f\|_p := \sup_{x\in X} \{p(f(x))\}$$

for each f in $B(X,E)$. When $E = \mathbb{K}$, in $B(X)$ it is define the norm of uniform convergence $\|\cdot\|$ as follows:

$$\|f\| := \sup_{x\in X} \{|f(x)|\}$$

for each f in $B(X)$.

Definition 1.2.2. *Let p be a seminorm of $s(E)$. In $B(X,E)$ it is define:*

- *the* ***p-uniform topology*** *as $u_p := \sigma(B(X,E), \{\|\cdot\|_p\})$ and*
- *the* ***uniform topology*** *as $\tau_u := \sigma(B(X,E), \{\|\cdot\|_q : q \in s(E)\})$.*

To denote the uniform topology when $E = \mathbb{K}$, we will use the symbol $\tau_{\|\cdot\|}$.

The collection of all compact subsets of X will be denoted by $K(X)$ and the collection of all finite subsets of X will be denoted by $A(X)$. We will denote by $\mathcal{P}$ a subset of $K(X)$ that satisfies the following conditions:

(a) $\mathcal{P}$ is a covering of X, that is to say, $X = \bigcup_{K\in\mathcal{P}} K$.

(b) $\mathcal{P}$ is a directed collection, this is, for each pair of elements K_1 and K_2 of $\mathcal{P}$ there exists an element K_3 in $\mathcal{P}$ such that $K_1 \cup K_2 \subset K_3$.

Definition 1.2.3. *Let p be a seminorm of $s(E)$ and K an element of $\mathcal{P}$. In $B(X,E)$ it is define the seminorm $\|\cdot\|_{p,K}$ as follows:*

$$\|f\|_{p,K} := \sup_{x\in K} \{p(f(x))\}$$

for each f in $B(X,E)$.

Definition 1.2.4. *Let p be a seminorm of $s(E)$. In $B(X,E)$ it is define:*

- *the **topology of p-uniform convergence in the elements of $\mathcal{P}$** as*

  $\tau_p := \tau_p(\mathcal{P}) := \sigma(B(X,E), \{\|\cdot\|_{p,K} : K \in \mathcal{P}\})$ *and*

- *the **topology of uniform convergence in the elements of $\mathcal{P}$** as*

  $\tau_{\mathcal{P}} := \sigma(B(X,E), \{\|\cdot\|_{q,K} : q \in s(E), K \in \mathcal{P}\})$.

Definition 1.2.5. *A function $v : X \longrightarrow [0,+\infty)$ **is $\mathcal{P}$-vanished at infinite** if it is of bounded range and if for each $\varepsilon > 0$, there exists K in $\mathcal{P}$ such that*

$$\|v\|_{X\setminus K} := \sup\{v(x) : x \in X\setminus K\} < \varepsilon.$$

The family of all functions $v : X \longrightarrow [0,+\infty)$ that $\mathcal{P}$-vanishes at infinite, will be denoted by $V_{\mathcal{P}}$. Since $\mathcal{P}$ is a directed set, it follows the following observation:

Remark 1.2.6. *Given v_1 and v_2 in $V_{\mathcal{P}}$ we have that $\text{máx}\{v_1, v_2\}$ belongs to $V_{\mathcal{P}}$.*

The fact that $\mathcal{P}$ is directed gives the following lemma that will be fundamental for the study of topologies that will be studied later.

Lemma 1.2.7. *Let (K_n) be a sequence of elements of $\mathcal{P}$ and $\sum_{n=1}^{\infty} a_n$ a convergent real non-negative numbers series. If (A_n) is a sequence of subsets of X, such that $A_n \subset K_n$ for each $n \in \mathbb{N}$, then the function $v = \sum_{n=1}^{\infty} a_n \mathscr{X}_{A_n} \in V_{\mathcal{P}}$.*

Proof. Given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\sum_{n=m+1}^{\infty} a_n < \varepsilon$. Since $\mathcal{P}$ is directed, we can choose $K \in \mathcal{P}$ such that $K_n \subset K$ for $n = 1, ..., m$. Thus $\|v\|_{X\setminus K} \leqslant \|\sum_{n=1}^{\infty} a_n \mathscr{X}_{K_n}\|_{X\setminus K} \leqslant \sum_{n=m+1}^{\infty} a_n < \varepsilon$. □

Definition 1.2.8. *Let p be a seminorm of $s(E)$ and v an element of $V_{\mathcal{P}}$. In $B(X,E)$ it is define the seminorm $\|\cdot\|_{p,v}$ as follows:*

$$\|f\|_{p,v} := \sup_{x\in X} \{v(x)p(f(x))\}$$

for each f in $B(X,E)$.

Definition 1.2.9. *Let p be a seminorm of $s(E)$. In $B(X,E)$ it is define the topologies:*

- $\beta_p := \beta_p(\mathcal{P}) := \sigma(B(X,E), \{\|\cdot\|_{p,v} : v \in V_{\mathcal{P}}\})$ *and*
- $\beta_{\mathcal{P}} := \sigma(B(X,E), \{\|\cdot\|_{q,v} : q \in s(E), v \in V_{\mathcal{P}}\})$.

Consider a family $\mathscr{R}$ of seminorms in a vector space V and $L \subset V$ a vector subspace. If p is in $\mathscr{R}$ then $p|_L$ will denote the restriction of p to the subspace L. Putting $\mathscr{R}|_L := \{p|_L : p \in \mathscr{R}\}$. It can be proved that in L the subspace topology induced by $\sigma(V,\mathscr{R})$ coincides with $\sigma(L, \mathscr{R}|_L)$. Thereby, in $C_b(X,E)$ the subspace topology induced by:

- u_p coincides with $\sigma\left(C_b(X,E), \{\|\cdot\|_p\}|_{C_b(X,E)}\right)$ which we will denote also by u_p.
- τ_u coincides with $\sigma\left(C_b(X,E), \{\|\cdot\|_q : q \in s(E)\}|_{C_b(X,E)}\right)$ which we will denote also by τ_u.
- τ_p coincides with $\sigma\left(C_b(X,E), \{\|\cdot\|_{p,K} : K \in \mathcal{P}\}|_{C_b(X,E)}\right)$ which we will denote also by τ_p.

- $\tau_{\mathcal{P}}$ coincides with $\sigma\left(C_b(X,E),\{\|\cdot\|_{q,K} : q\in s(E), K\in\mathcal{P}\}\,|_{C_b(X,E)}\right)$ which we will denote also by $\tau_{\mathcal{P}}$.

- β_p coincides with $\sigma\left(C_b(X,E),\{\|\cdot\|_{p,v} : v\in V_{\mathcal{P}}\}\,|_{C_b(X,E)}\right)$ which we will denote also by β_p.

- $\beta_{\mathcal{P}}$ coincides with $\sigma\left(C_b(X,E),\{\|\cdot\|_{q,v} : q\in s(E), v\in V_{\mathcal{P}}\}\,|_{C_b(X,E)}\right)$ which we will denote also by $\beta_{\mathcal{P}}$.

Let $p\in s(E)$. Since $s(E)$ is a family of directed seminorms and $\mathcal{P}$ is a directed set, it can be proved that the family of seminorms $\{\|\cdot\|_q : q\in s(E)\}$, $\{\|\cdot\|_{p,K} : K\in\mathcal{P}\}$, $\{\|\cdot\|_{q,K} : q\in s(E), K\in\mathcal{P}\}$, $\{\|\cdot\|_{p,v} : v\in V_{\mathcal{P}}\}$ and $\{\|\cdot\|_{q,v} : q\in s(E), v\in V_{\mathcal{P}}\}$ are directed. To prove this in the last two, the observation 1.2.6 is used. Thereby we can state the following proposition:

Proposition 1.2.10. *Each locally convex topologies $\tau_{_E}, u_p, \tau_u, \tau_p, \tau_{\mathcal{P}}, \beta_p$ and $\beta_{\mathcal{P}}$ have a base of neighborhoods of 0 of the type given in the proposition 1.1.2.*

Remark 1.2.11.

- *In $C_b(X,E)$ the topologies of the type $\beta_{\mathcal{P}}$ are called strict topologies.*

- *To simplify the notations, we will accept the following convention: if $\|\cdot\|_\psi$ is a seminorm in $B(X,E)$, then, to refer to its restriction to the space $C_b(X,E)$, we simply write $\|\cdot\|_\psi$.*

- *When $E=\mathbb{K}$, we will write $\gamma_{\mathcal{P}}$ instead of $\beta_{\mathcal{P}}$.*

- *When $\mathcal{P}=A(X)$, we will write δ instead of $\beta_{_{A(X)}}$.*

- *When $\mathcal{P}=K(X)$, we will write $\beta_\circ$ instead of $\beta_{_{K(X)}}$ and $\gamma_\circ$ instead of $\gamma_{_{K(X)}}$ (in the case $E=\mathbb{K}$).*

- *The topologies of pointwise convergence in $B(X,E)$ and $C_b(X,E)$ will be both denoted by pw.*

1.3. Baire Measures.

Throughout the thesis, we will assume the knowledge of the basic concepts of measure theory; for example, the definition of a set algebra, a monotone sets function and a finitely additive function.

1.3.1. Basic Concepts.

In the following, we will present some definitions and preliminary results that appears in [7], and which are necessary to develop the concepts of Baire measures and Baire sets. During this section, some proofs will be omitted since they are already explained in [7].

Let $\mathcal{X}$ be a set, Σ be a set algebra of $\mathcal{X}$ and μ be a set function with domain in Σ and values in the extended real numbers set $\mathbb{R}_e$.

Definition 1.3.1.1. *The* ***total variation of $\boldsymbol{\mu}$*** *is the function $|\mu| : \Sigma \to \mathbb{R}_e$, defined by*

$$|\mu|(A) := \sup \left\{ \sum_{i=1}^{n} |\mu(A_i)| : \{A_i\}_{i=1}^{n} \textit{ is a } \Sigma\textit{-partition of } A \right\} .$$

Remark 1.3.1.2. *If $\mu : \Sigma \to \mathbb{R}$ is a set function, then the composition of μ with the absolute value will be denoted by $|\mu(\cdot)|$ and shall not be confused with $|\mu|$ which is the notation of total variation of μ.*

Definition 1.3.1.3. *We say $\mu : \Sigma \to \mathbb{R}_e$ is* ***bounded*** *if* $\sup \{|\mu(A)| : A \in \Sigma\} < \infty$.

Lemma 1.3.1.4. *If $\mu : \Sigma \to \mathbb{R}$ is finitely additive and bounded, then,*

$$|\mu|(\mathcal{X}) \leqslant 4 \sup \{|\mu(A)| : A \in \Sigma\} .$$

Lemma 1.3.1.5. *If μ is non-negative and finitely additive, then, $\mu = |\mu|$.*

Lemma 1.3.1.6. *If μ is a finitely additive set function, then, so $|\mu|$ is.*

Lemma 1.3.1.7. *If $\mu : \Sigma \to \mathbb{R}$ is a non-negative and finitely additive set function, then, it is monotone and bounded. In fact, we have:*

$$\mu(\mathcal{X}) = \sup\{\mu(A) : A \in \Sigma\}\,.$$

Proof. Let $A, B \in \Sigma$ be $A \subset B$. Since μ is finitely additive and non-negative, we have: $\mu(B) = \mu(A \cup (B \backslash A)) = \mu(A) + \mu(B \backslash A) \geqslant \mu(A)$. That is, μ is monotone, since for each $A \in \Sigma$, we have $\mu(A) \leqslant \mu(\mathcal{X})$, and, as a consequence, $\mu(\mathcal{X}) = \sup\{\mu(A) : A \in \Sigma\}$. □

Definition 1.3.1.8. *Let $\mu : \Sigma \to \mathbb{R}$ be a bounded and finitely additive set function. The **positive variation of $\boldsymbol{\mu}$** and the **negative variation of $\boldsymbol{\mu}$**, are functions denoted and defined, respectively, by:*

$$\mu^+ : \Sigma \to \mathbb{R}, \quad \mu^+ := \frac{1}{2}(|\mu| + \mu) \qquad \mu^- : \Sigma \to \mathbb{R}, \quad \mu^- := \frac{1}{2}(|\mu| - \mu).$$

Note that μ^+ and μ^- are bounded, non-negative and finitely additive functions.

Theorem 1.3.1.9. *(Jordan decomposition theorem.) If $\mu : \Sigma \to \mathbb{R}$ is bounded and finitely additive, then for each $A \in \Sigma$, we have:*

$$\mu^+(A) = \sup\{\mu(F) : F \in \Sigma, F \subset A\} \quad \textit{and}$$
$$\mu^-(A) = -\inf\{\mu(F) : F \in \Sigma, F \subset A\}\ .$$

Also, $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$ are satisfied.

Corollary 1.3.1.10. *Let μ, μ_1, μ_2 be three real-valued set functions on Σ. If μ, μ_1, μ_2 are bounded, finitely additive, non-negative and $\mu = \mu_1 - \mu_2$, then, $\mu^+ \leqslant \mu_1$ and $\mu^- \leqslant \mu_2$.*

Proof. Note that $-\mu_2 \leqslant \mu \leqslant \mu_1$. By lemma 1.3.1.7, μ_1 and μ_2 are monotone measures. Then, for arbitrary $A \in \mathcal{B}$, we have:

$$\mu^+(A) = \sup\{\mu(F) : F \in \Sigma, F \subset A\} \leqslant \sup\{\mu_1(F) : F \in \Sigma, F \subset A\} = \mu_1(A)$$
$$\mu^-(A) = \sup\{-\mu(F) : F \in \Sigma, F \subset A\} \leqslant \sup\{\mu_2(F) : F \in \Sigma, F \subset A\} = \mu_2(A)$$

□

1.3.2. Baire Sets.

Definition 1.3.2.1. *We denote by $\mathcal{Z}$ the collection:*

$$\mathcal{Z} := \left\{ f^{-1}(\{0\}) : f \in C_b(X) \right\} .$$

The elements of $\mathcal{Z}$ are called zero sets.

Lemma 1.3.2.2. *Finite unions and countable intersections of elements of $\mathcal{Z}$, are also on $\mathcal{Z}$.*

Proof. Let $Z_1, \ldots, Z_n$ be elements of $\mathcal{Z}$. For each $i \in \{1, \ldots, n\}$, we take $f_i \in C_b(X)$ such that $Z_i = f_i^{-1}(\{0\})$. The proof is done since $Z_1 \cup \ldots \cup Z_n = (f_1 \cdot \ldots \cdot f_n)^{-1}(\{0\})$.

Let us consider a sequence $(Z_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{Z}$. For each $n \in \mathbb{N}$, we take $f_n \in C_b(X)$ such that $Z_n = f_n^{-1}(\{0\})$. If we define the function $g : X \to \mathbb{R}$, by $g(x) = \sum_{n=1}^{\infty} |f_n(x)|/2^n$, then, it is clear that $\bigcap_{n=1}^{\infty} Z_n = g^{-1}(\{0\})$. □

Definition 1.3.2.3. *We denote by $\mathcal{U}$ the collection:*

$$\mathcal{U} := \{U \subset X : X \backslash U \in \mathcal{Z}\} .$$

The elements of $\mathcal{U}$ are called co-zero sets.

Directly from the definition of $\mathcal{U}$ and the previous lemma, we have the following result.

Lemma 1.3.2.4. *A finite intersection and a countable union of elements of $\mathcal{U}$, are elements of $\mathcal{U}$.*

Lemma 1.3.2.5. *Suppose that $\mathbb{K} = \mathbb{R}$. Let $a \in \mathbb{R}$ and $f \in C_b(X)$, the following statements are satisfied:*

(a) $\{x \in X : f(x) \leqslant a\} \in \mathcal{Z}$.

(b) $\{x \in X : f(x) \geqslant a\} \in \mathcal{Z}$.

(c) $\{x \in X : f(x) < a\} \in \mathcal{U}$.

(d) $\{x \in X : f(x) > a\} \in \mathfrak{U}$.

Proof. (a) Let $a \in \mathbb{R}$. Considering $g = \text{máx}\{f, a\} - a$. We have $g \in C_b(X)$ and $g^{-1}(\{0\}) = \{x \in X : \text{máx}\{f, a\} = a\} = \{x \in X : f(x) \leqslant a\} \in \mathcal{Z}$.

(b) Let $a \in \mathbb{R}$. Considering $g = \text{mín}\{f, a\} - a$. We have $g \in C_b(X)$ and $g^{-1}(\{0\}) = \{x \in X : \text{mín}\{f, a\} = a\} = \{x \in X : f(x) \geqslant a\} \in \mathcal{Z}$.

(c) It is a direct consequence of (b).

(d) It is a direct consequence of (a). □

Lemma 1.3.2.6. *If $Z_1, Z_2 \in \mathcal{Z}$, are such that, $Z_1 \cap Z_2 = \emptyset$, then, there exists $f \in C_b(X)$, such that $0 \leq f \leq \mathscr{X}_X$, $Z_1 = f^{-1}(\{0\})$ and $Z_2 = f^{-1}(\{1\})$.*

Proof. Let $g_1, g_2 \in C_b(X)$, such that $Z_i = g_i^{-1}(\{0\})$ for $i = 1, 2$. The function $f = g_1^2/(g_1^2 + g_2^2)$, satisfies the requirements. □

A function f given in the previous lemma, will be called $\boldsymbol{Z_1}$ **and** $\boldsymbol{Z_2}$ **connection**.

Lemma 1.3.2.7. *If $\{Z_i\}_{i=1}^n \subset \mathcal{Z}$ and $\{U_i\}_{i=1}^n \subset \mathfrak{U}$ are such that, $n > 1$, $Z_i \cap Z_j = \emptyset$, if $i \neq j$ and $Z_i \subset U_i$ for each $i \in \{1, \ldots, n\}$, then, there exist collections $\{C_i\}_{i=1}^n \subset \mathcal{Z}$ and $\{V_i\}_{i=1}^n \subset \mathfrak{U}$ such that, for $i, j \in \{1, \ldots, n\}$, $i \neq j$, $C_i \cap C_j = \emptyset$ and $Z_i \subset V_i \subset C_i \subset U_i$.*

Proof. By lemmas 1.3.2.2 and 1.3.2.6, we can consider a connection $f_1 \in C_b(X)$ of $(X \backslash U_1) \cup \bigcup_{i=2}^n Z_i$ and Z_1. Putting $V_1 := f_1^{-1}((\frac{1}{2}, +\infty))$, we have $Z_1 \subset V_1$ and

$$(X \backslash U_1) \cup \bigcup_{i=2}^n Z_i = f_1^{-1}(\{0\}) \subset f_1^{-1}((-\infty, 1/2]) = X \backslash V_1\,,$$

therefore, $V_1 \cap Z_i = \emptyset$ for $i = 2, \ldots, n$, and $Z_1 \subset V_1 \subset U_1$.

Since $V_1 \subset C_1 := f_1^{-1}(\left[\frac{1}{2}, +\infty\right))$ and $f_1^{-1}(\{0\}) \cap C_1 = \emptyset$, we have $C_1 \cap Z_i = \emptyset$, for $i = 2, \ldots, n$ and $C_1 \subset U_1$. In summary, we have $Z_1 \subset V_1 \subset C_1 \subset U_1$ and by lemma 1.3.2.5, $V_1 \in \mathfrak{U}$ and $C_1 \in \mathcal{Z}$.

Then, for $k \in \{2, \ldots, n\}$, we take recursively connections $f_k \in C_b(X)$ of $(X \backslash U_k) \cup \bigcup_{\substack{i=1 \\ i \neq k}}^n Z_i \cup \bigcup_{j=1}^{k-1} C_j$ and Z_k. Putting $V_k := f_k^{-1}((\frac{1}{2}, +\infty))$, we have $Z_k \subset V_k$ and

$f_k^{-1}(\{0\}) \subset f_k^{-1}\big((-\infty, \frac{1}{2}]\big) \subset X \backslash V_k$. Therefore, $Z_k \subset V_k \subset U_k$. Since $V_k \subset C_k := f_k^{-1}\big([\frac{1}{2}, +\infty)\big)$ and $f_k^{-1}(\{0\}) \cap C_k = \emptyset$, we have $C_k \cap C_i = \emptyset$ for $i = 1, \ldots, k-1$ and $C_k \subset U_k$. In summary, we have $Z_k \subset V_k \subset C_k \subset U_k$, $C_k \cap C_i = \emptyset$ for $i = 1, \ldots, k-1$ and by lemma 1.3.2.5, $V_k \in \mathcal{U}$ and $C_k \in \mathcal{Z}$.

Therefore, the collections $\{C_i\}_{i=1}^n$ and $\{V_i\}_{i=1}^n$ are the required finite sequence. □

Definition 1.3.2.8. *The **Baire algebra** is defined as the smaller algebra of subsets of X which contains $\mathcal{Z}$ (or $\mathcal{U}$). This is denoted by $\mathcal{B}$ and its elements are called **Baire subsets**.*

1.3.3. Baire measures space.

Definition 1.3.3.1. *A set function $\mu : \mathcal{B} \to \mathbb{R}$ is **regular**, if for each $A \in \mathcal{B}$, we have: $\mu(A) = \sup\{\mu(Z) : Z \in \mathcal{Z},\ \ Z \subset A\}$.*

Definition 1.3.3.2. *A **positive Baire measure** is a non-negative, finitely additive and regular set function $\mu : \mathcal{B} \to \mathbb{R}$. The set of all positive Baire measures are denoted by $M^+(X)$.*

*The difference $\mu_1 - \mu_2$ between two positive Baire measures, μ_1 and μ_2, is called **Baire measure**. The set of all Baire measures is denoted by $M(X)$.*

Proposition 1.3.3.3. *Let $\mu : \Sigma \to \mathbb{R}$ be a non-negative and finitely additive set function. The following statements are equivalent:*

(a) $\mu \in M^+(X)$.

(b) *For each $A \in \mathcal{B}$, $\mu(A) = \text{ínf}\{\mu(U) : U \in \mathcal{U},\ \ A \subset U\}$.*

Proof. Suppose that $\mu \in M^+(X)$. Then, by the regularity of μ, for each $A \in \mathcal{B}$, we have:

$$\begin{aligned}\mu(X\backslash A) &= \sup\{\mu(Z) : Z \in \mathcal{Z},\ \ Z \subset X\backslash A\}\\ &= \sup\{\mu(X\backslash(X\backslash Z)) : Z \in \mathcal{Z},\ \ Z \subset X\backslash A\}\\ &= \mu(X) + \sup\{-\mu(X\backslash Z) : Z \in \mathcal{Z},\ \ Z \subset X\backslash A\}\\ &= \mu(X) - \text{ínf}\{\mu(U) : U \in \mathcal{U},\ \ A \subset U\}\end{aligned}$$

Thus, $\text{ínf}\{\mu(U) : U \in \mathcal{U},\ \ A \subset U\} = \mu(X) - \mu(X\backslash A) = \mu(A)$.

Now, supposed that (b) is satisfied. For each $A \in \mathcal{B}$, we have:

$$\begin{aligned}\mu(X\backslash A) &= \text{ínf}\{\mu(U) : U \in \mathcal{U},\ \ X\backslash A \subset U\}\\ &= \text{ínf}\{\mu(X\backslash(X\backslash U)) : U \in \mathcal{U},\ \ X\backslash U \subset A\}\\ &= \mu(X) + \text{ínf}\{-\mu(X\backslash U) : U \in \mathcal{U},\ \ X\backslash U \subset A\}\\ &= \mu(X) - \sup\{\mu(Z) : Z \in \mathcal{Z},\ \ Z \subset A\}\end{aligned}$$

Then, we have: $\sup\{\mu(Z) : Z \in \mathcal{Z},\ \ Z \subset A\} = \mu(X) - \mu(X\backslash A) = \mu(A)$. Therefore, μ is regular and also $\mu \in M^+(X)$. □

Proposition 1.3.3.4. *If $\mu_1, \mu_2 \in M^+(X)$ and $\alpha, \beta \geqslant 0$, then, $\alpha\mu_1 + \beta\mu_2 \in M^+(X)$.*

Proof. Let $A \in \mathcal{B}$ and $\varepsilon > 0$. There exist, $Z_1, Z_2 \in \mathcal{Z}$ such that:

$$\begin{aligned}(\alpha\mu_1 + \beta\mu_2)(A) &= \alpha\mu_1(A) + \beta\mu_2(A)\\ &= \alpha \sup\{\mu_1(Z) : Z \in \mathcal{Z},\ \ Z \subset A\} + \beta \sup\{\mu_2(Z) : Z \in \mathcal{Z},\ \ Z \subset A\}\\ &\leqslant \alpha(\mu_1(Z_1) + \varepsilon) + \beta(\mu_2(Z_2) + \varepsilon)\\ &= \alpha\mu_1(Z_1) + \beta\mu_2(Z_2) + \varepsilon(\alpha + \beta)\\ &\leqslant \alpha\mu_1(Z_1 \cup Z_2) + \beta\mu_2(Z_1 \cup Z_2) + \varepsilon(\alpha + \beta)\\ &\leqslant \sup\{(\alpha\mu_1 + \beta\mu_2)(Z) : Z \in \mathcal{Z},\ \ Z \subset A\} + \varepsilon(\alpha + \beta)\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can conclude that:

$$(\alpha\mu_1 + \beta\mu_2)(A) = \sup\{(\alpha\mu_1 + \beta\mu_2)(Z) : Z \in \mathcal{Z},\ \ Z \subset A\}\,.$$

Therefore, $\alpha\mu_1 + \beta\mu_2 \in M^+(X)$. □

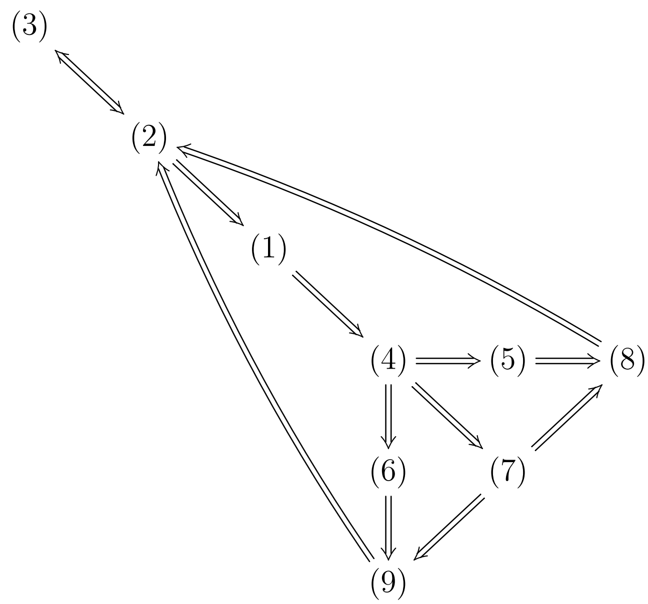
The following corollary is an immediate consequence from the previous Proposition.

Corollary 1.3.3.5. *The Baire measures set $M(X)$ is a real vector space.*

Theorem 1.3.3.6. *Let $\mu : \Sigma \to \mathbb{R}$ be a bounded and finitely additive sets function. The following statements are equivalent:*

1. $\mu \in M(X)$
2. $\mu^+, \mu^- \in M^+(X)$
3. $|\mu| \in M(X)$
4. *Given $A \in \mathcal{B}$ and $\varepsilon > 0$, there exist $Z \in \mathcal{Z}$ and $U \in \mathcal{U}$ such that $Z \subset A \subset U$ and $|\mu|(B) < \varepsilon$, $\forall B \in \mathcal{B}$, $B \subset U \backslash Z$.*
5. *Given $A \in \mathcal{B}$ and $\varepsilon > 0$, there exists $Z \in \mathcal{Z}$ such that $Z \subset A$ and $|\mu|(B) < \varepsilon$, $\forall B \in \mathcal{B}$, $B \subset A \backslash Z$.*
6. *Given $A \in \mathcal{B}$ and $\varepsilon > 0$, there exists $U \in \mathcal{U}$ such that $A \subset U$ and $|\mu|(B) < \varepsilon$, $\forall B \in \mathcal{B}$, $B \subset U \backslash A$.*
7. *Given $A \in \mathcal{B}$ and $\varepsilon > 0$, there exist $Z \in \mathcal{Z}$ and $U \in \mathcal{U}$ such that $Z \subset A \subset U$ and $|\mu(B)| < \varepsilon$, $\forall B \in \mathcal{B}$, $B \subset U \backslash Z$.*
8. *Given $A \in \mathcal{B}$ and $\varepsilon > 0$, there exists $Z \in \mathcal{Z}$ such that $Z \subset A$ and $|\mu(B)| < \varepsilon$, $\forall B \in \mathcal{B}$, $B \subset A \backslash Z$.*
9. *Given $A \in \mathcal{B}$ and $\varepsilon > 0$, there exists $U \in \mathcal{U}$ such that $A \subset U$ and $|\mu(B)| < \varepsilon$, $\forall B \in \mathcal{B}$, $B \subset U \backslash A$.*

Proof. The strategy of the proof is shown in the diagram below.



The proofs of (2) $\Rightarrow$ (1), (4) $\Rightarrow$ (7), (7) $\Rightarrow$ (8), (7) $\Rightarrow$ (9), (4) $\Rightarrow$ (5) $\Rightarrow$ (8) and (4) $\Rightarrow$ (6) $\Rightarrow$ (9) are immediate.

We will show that (1) $\Rightarrow$ (4). Suppose that $\mu \in M(X)$. Let $\mu_1, \mu_2 \in M^+(X)$ be such that $\mu = \mu_1 - \mu_2$. Let $A \in \mathcal{B}$ and $\varepsilon > 0$. By the regularity of μ_i ($i = 1, 2$) and by 1.3.3.3, there exist $Z_1, Z_2 \in \mathcal{Z}$ and $U_1, U_2 \in \mathcal{U}$ such that $Z_i \subset A \subset U_i$ and $\mu_i(U_i) - \frac{\varepsilon}{4} < \mu_i(A) < \mu_i(Z_i) + \frac{\varepsilon}{4}$, for $i = 1, 2$. Then, $\mu_i(U_i \backslash Z_i) < \frac{\varepsilon}{2}$, for $i = 1, 2$. Taking $U = U_1 \cap U_2$ and $Z = Z_1 \cup Z_2$ we have $Z \in \mathcal{Z}$, $U \in \mathcal{U}$, $Z \subset A \subset U$ and $\mu_i(U \backslash Z) \leqslant \mu_i(U_i \backslash Z_i) < \frac{\varepsilon}{2}$, for $i = 1, 2$. If $B \in \mathcal{B}$, $B \subset U \backslash Z$, then applying 1.3.1.10, we have: $|\mu|(B) = \mu^+(B) + \mu^-(B) \leqslant \mu_1(B) + \mu_2(B) \leqslant \mu_1(U \backslash Z) + \mu_2(U \backslash Z) < \varepsilon$. Thus, (4) is satisfied.

We will show that (8) $\Rightarrow$ (2). By hypothesis, given $A \in \mathcal{B}$ and $\varepsilon > 0$, there exists $Z \in \mathcal{Z}$, $Z \subset A$, such that $-\varepsilon < \mu(B) < \varepsilon$, for all $B \in \mathcal{B}$, $B \subset A \backslash Z$. Then,

$$\mu^+(A \backslash Z) = \sup\{\mu(B) : B \in \mathcal{B}, B \subset A \backslash Z\} < \varepsilon \quad \text{and}$$
$$-\mu^-(A \backslash Z) = \text{ínf}\{\mu(B) : B \in \mathcal{B}, B \subset A \backslash Z\} > -\varepsilon.$$

Thus,

$$\mu^+(A) < \varepsilon + \mu^+(Z) \leqslant \varepsilon + \sup\{\mu^+(G) : G \in \mathcal{Z}, G \subset A\} \quad \text{and}$$
$$\mu^-(A) < \varepsilon + \mu^-(Z) \leqslant \varepsilon + \sup\{\mu^-(G) : G \in \mathcal{Z}, G \subset A\}.$$

Since $\varepsilon > 0$ is arbitrary and μ^+, μ^- are monotone, it is concluded that: $\mu^+(A) = \sup\{\mu^+(G) : G \in \mathcal{Z}, G \subset A\}$ and $\mu^-(A) = \sup\{\mu^-(G) : G \in \mathcal{Z}, G \subset A\}$. In other words, $\mu^+, \mu^- \in M^+(X)$.

We will show that (9) $\Rightarrow$ (2). By hypothesis, given $A \in \mathcal{B}$ and $\varepsilon > 0$, there exists $U \in \mathcal{U}$ such that $A \subset U$ and $-\varepsilon < \mu(B) < \varepsilon$, for all $B \in \mathcal{B}$, $B \subset U \backslash A$. Then,

$$\mu^+(U\backslash A) = \sup\{\mu(B) : B \in \mathcal{B}, B \subset U\backslash A\} < \varepsilon \quad \text{and}$$
$$-\mu^-(U\backslash A) = \text{ínf}\{\mu(B) : B \in \mathcal{B}, B \subset U\backslash A\} \geqslant -\varepsilon.$$

Thus,

$$\text{ínf}\{\mu^+(G) : G \in \mathcal{U}, A \subset G\} \leqslant \mu^+(U) < \varepsilon + \mu^+(A) \quad \text{and}$$
$$\text{ínf}\{\mu^-(G) : G \in \mathcal{U}, A \subset G\} \leqslant \mu^-(U) < \varepsilon + \mu^-(A).$$

Again, since $\varepsilon > 0$ is arbitrary and μ^+, μ^- are monotone, it is concluded that: $\mu^+(A) = \text{ínf}\{\mu^+(G) : G \in \mathcal{U}, A \subset G\}$ and $\mu^-(A) = \text{ínf}\{\mu^-(G) : G \in \mathcal{U}, A \subset G\}$. In other words, $\mu^+, \mu^- \in M^+(X)$.

Let us prove now (3) $\Rightarrow$ (2). Suppose that $|\mu| \in M^+(X)$ and let $A \in \mathcal{B}$.

$$\begin{aligned}\mu^+(A) + \mu^-(A) = |\mu|(A) =& \sup\{|\mu|(Z) : G \in \mathcal{Z}, Z \subset A\} \\ &\leqslant \sup\{\mu^+(Z) : Z \in \mathcal{Z}, Z \subset A\} + \sup\{\mu^-(Z) : Z \in \mathcal{Z}, Z \subset A\}\ .\end{aligned}$$

Reasoning by contradiction, if $\mu^+(A) > \sup\{\mu^+(Z) : Z \in \mathcal{Z}, Z \subset A\}$, then $0 < \mu^+(A) - \sup\{\mu^+(Z) : Z \in \mathcal{Z}, Z \subset A\} \leqslant \sup\{\mu^-(Z) : Z \in \mathcal{Z}, Z \subset A\} - \mu^-(A)$. Then, $\mu^-(A) < \sup\{\mu^-(Z) : Z \in \mathcal{Z}, Z \subset A\} \leqslant \mu^-(A)$, which is a contradiction. Thus, $\mu^+(A) = \sup\{\mu^+(Z) : Z \in \mathcal{Z}, Z \subset A\}$ and $\mu^-(A) = \sup\{\mu^-(Z) : Z \in \mathcal{Z}, Z \subset A\}$. Therefore, μ^+ and μ^- belong to $M^+(X)$.

Finally, in order to prove (2) $\Rightarrow$ (3), note that if $\mu^+, \mu^- \in M^+(X)$, then by Proposition 1.3.3.4, $|\mu| = \mu^+ + \mu^- \in M^+(X)$. □

Corollary 1.3.3.7. *Let $\mu : \mathcal{B} \to \mathbb{R}$ be a set function. If $\mu \in M(X)$ and $\mu \geqslant 0$, then, $\mu^- = 0$ and $\mu = |\mu| = \mu^+ \in M^+(X)$.*

Proof. Since μ is non-negative and finitely additive measure, we have $\mu = |\mu|$. From this, $\mu^- = 0$ and then $\mu = \mu^+$. Thus, applying the previous theorem, we conclude that $|\mu| = \mu^+ \in M^+(X)$. □

Definition 1.3.3.8. *The map* $\|\cdot\|_*$ *defined by* $\|\mu\|_* := |\mu|(X)$ *is a norm in* $M(X)$*.*

Chapter 2

Some topological properties of $(C_b(X,E), \beta_{\mathcal{P}})$.

All the results in chapter 2 will be established and proved for the space $C_b(X,E)$. However, in the sections 2.1, 2.2 and 2.3 it is enough to replace the symbol $C_b(X,E)$ by $B(X,E)$ to make the results and proofs equally valid in the space $B(X,E)$.

2.1. Comparison of topologies in $C_b(X,E)$ and $B(X,E)$.

In this section we will see that the different topologies that we have defined in the space $C_b(X,E)$ (and in $B(X,E)$) are comparable, and we will indicate which topologies are finer than others and what conditions are necessary and sufficient to obtain the equality between them.

Definition 2.1.1. *Let L_1 and L_2 be two coverings of X. We will say that L_2 is a* ***refinement*** *of L_1 if each element of L_2 is contained in some element of L_1. We will denote this by $L_1 \preceq L_2$.*

Remark 2.1.2. *Note that the collections $A(X), K(X)$ and $\mathcal{P}$, satisfy:*

$$K(X) \preceq \mathcal{P} \preceq A(X).$$

Remark 2.1.3. *If $X \in \mathcal{P}$, then, $\mathcal{P} \preceq K(X)$.*

Proposition 2.1.4.

(a) If $\mathcal{P}_2 \preceq \mathcal{P}_1$ then $\tau_{\mathcal{P}_1} \leqslant \tau_{\mathcal{P}_2}$ and $\beta_{\mathcal{P}_1} \leqslant \beta_{\mathcal{P}_2}$

(b) $\tau_{\mathcal{P}} \leqslant \beta_{\mathcal{P}} \leqslant \tau_u$.

(c) $\tau_{\mathcal{P}} = \beta_{\mathcal{P}}$ if and only if, for each countable union of elements of $\mathcal{P}$, there exists an element of $\mathcal{P}$ containing this union.

(d) $\beta_{\mathcal{P}} = \tau_u$ if and only if $X \in \mathcal{P}$.

Proof. (a) If $\mathcal{P}_1$ is a refinement of $\mathcal{P}_2$, then, given K_1 in $\mathcal{P}_1$ there exists K_2 in $\mathcal{P}_2$ containing K_1. Then, the first relation follows the fact that $\|\cdot\|_{K_1,p} \leqslant \|\cdot\|_{K_2,p}$ for each $p \in s(E)$. To see that $\beta_{\mathcal{P}_1} \leqslant \beta_{\mathcal{P}_2}$, it is enough to show that $V_{\mathcal{P}_1} \subset V_{\mathcal{P}_2}$. Let $v \in V_{\mathcal{P}_1}$ and $\varepsilon > 0$. Then, there exists $K_1 \in \mathcal{P}_1$ for which $\|v\|_{X\setminus K_1} < \varepsilon$ and also, by hypothesis, there exists K_2 in $\mathcal{P}_2$, containing K_1. Therefore, $\|v\|_{X\setminus K_2} \leqslant \|v\|_{X\setminus K_1} < \varepsilon$, and then $v \in V_{\mathcal{P}_2}$.

(b) Let $p \in s(E)$ and $K \in \mathcal{P}$. Let us put $v = \mathscr{X}_K$. Then, $\|\cdot\|_{p,K} = \|\cdot\|_{p,v}$. From this $\tau_{\mathcal{P}} \leqslant \beta_{\mathcal{P}}$ since the family of seminorms which generates $\tau_{\mathcal{P}}$ is contained in the family of seminorms which generates $\beta_{\mathcal{P}}$. Furthermore, given v in $V_{\mathcal{P}}$, p in $\mathcal{P}$ and f in $C_b(X,E)$, we have $\|f\|_{p,v} \leqslant \|v\|\|f\|_p$. therefore $\beta_{\mathcal{P}} \leqslant \tau_u$.

(c) Suppose that $\tau_{\mathcal{P}}$ and $\beta_{\mathcal{P}}$ coincide and prove that for any sequence (K_n) of elements of $\mathcal{P}$, there exists $K \in \mathcal{P}$. such that $\bigcup_{n=1}^{\infty} K_n \subset K$. Let (K_n) be a sequence of elements of $\mathcal{P}$. Applying 1.2.7, the map $v = \sum_{n=1}^{\infty} 2^{-n} \mathscr{X}_{K_n}$ is in $V_{\mathcal{P}}$. By hypothesis, for each $p \in$s(E) , there exist $q \in$s(E) , $K \in \mathcal{P}$. and $\varepsilon > 0$ such that:

$$\{f \in C_b(X,E) : \|f\|_{q,K} < \varepsilon\} \subset \{f \in C_b(X,E) : \|f\|_{p,v} < 1\}\,.$$

Then, if $f \in C_b(X,E)$ is such that $\|f\|_{q,K} \neq 0$, then $\|\,(\varepsilon/2\|f\|_{q,K})\,f\|_{p,v} < 1$ which implies that $\|f\|_{p,v} < (2/\varepsilon)\|f\|_{q,K}$. On the other hand, if $\|f\|_{q,K} = 0$, then $\|nf\|_{q,K} = 0$ for all $n \in \mathbb{N}$ and since $\|nf\|_{p,v} < 1$ equivalently $\|f\|_{p,v} = 0$. As a consequence, we have proved that $\|\cdot\|_{p,v} \leqslant r\|\cdot\|_{q,K}$ where $r = 2/\varepsilon$.

We will show by contradiction that $\forall n \in \mathbb{N}, K_n \subset K$. Suppose $K_n \not\subset K$ for some $n \in \mathbb{N}$. Since X is completely regular, for $y \in K_n \backslash K$ there exists $g \in C_b(X)$ such that $0 \leqslant g \leqslant \mathscr{X}_X$, $g(K) = \{0\}$ and $g(y) = 1$. Choose $s \in E$ such that $p(s) \neq 0$ and consider the function $g \otimes s \in C_b(X,E)$. As we obtained above $\|g \otimes s\|_{p,v} \leqslant r\|g \otimes s\|_{q,K} = 0$. However, for $y \in K_n \backslash K$, $0 < v(y)p(s) \leqslant \|g \otimes s\|_{p,v}$ which is a contradiction. Therefore $\bigcup_{n=1}^{\infty} K_n \subset K$.

Conversely, suppose that for each sequence (K_n) of elements of $\mathcal{P}$, there exists $K \in \mathcal{P}$ such that for all $n \in \mathbb{N}$, $K_n \subset K$. We will prove that $\beta_{\mathcal{P}} \leqslant \tau_{\mathcal{P}}$. Let $v \in V_{\mathcal{P}}$ and $p \in s(E)$. It is know that for each $n \in \mathbb{N}$, there exists $K_n \in \mathcal{P}$ such that $\|v\|_{X\backslash K_n} < 2^{-n}$. By hypothesis, there exists a K set in $\mathcal{P}$ such that $\bigcup_{n=1}^{\infty} K_n \subset K$, therefore $\|v\|_{X\backslash K} \leqslant \|v\|_{X\backslash K_n}$ for each $n \in \mathbb{N}$. Which implies that for each $x \in X\backslash K$, $v(x) = 0$.

Then, for $f \in C_b(X,E)$, $\|f\|_{p,v} = \sup\{v(x)p(f(x)) : x \in K\} \leqslant \|v\|\|f\|_{p,K}$. Thus it is proved that $\beta_{\mathcal{P}} \leqslant \tau_{\mathcal{P}}$.

(d) Suppose that $\tau_u \leqslant \beta_{\mathcal{P}}$. Then, for $p \in s(E)$, there exist $q \in s(E)$, $v \in V_{\mathcal{P}}$ and $r > 0$ such that $\|\cdot\|_p \leqslant r\|\cdot\|_{q,v}$. Evaluating constant functions in the previous inequality, it follows that $p(e) \leqslant r\|v\|q(e)$ for each $e \in E$. Let $s \in E$ such that $p(s) = 1$.

There exists $K \in \mathcal{P}$ for which $v(x) \leqslant 1/2rq(s)$, $x \in X\backslash K$. which implies $X\backslash K = \emptyset$ and then $X \in \mathcal{P}$;. In fact, otherwise we can choose $x \in X\backslash K$ and $f \in C_b(X)$ such that $0 \leqslant f \leqslant \mathscr{X}_X$, $f(x) = 1$ and $f(K) = \{0\}$ but this is a contradiction, since:

$$r\|f \otimes s\|_{q,v} = r \sup_{x \in X\backslash K}\{v(x)q(f(x)s)\} \leqslant \frac{rq(s)}{2rq(s)} = \frac{1}{2} < 1 = \|f \otimes s\|_p.$$

Conversely, if $X \in \mathcal{P}$, then $\mathscr{X}_X \in V_{\mathcal{P}}$. This implies that $\|\cdot\|_p = \|\cdot\|_{p,\mathscr{X}_X}$ for each $p \in s(E)$. Thus, the family of seminorms which generates the topology τ_u is contained in the family of seminorms which generates the topology $\beta_{\mathcal{P}}$. □

Analogously, we have the following proposition:

Proposition 2.1.5. *Let $p \in s(E)$.*

(a) If $\mathcal{P}_2 \preceq \mathcal{P}_1$ then $\tau_p(\mathcal{P}_1) \leqslant \tau_p(\mathcal{P}_2)$ and $\beta_p(\mathcal{P}_1) \leqslant \beta_p(\mathcal{P}_2)$.

(b) $\tau_p \leqslant \beta_p \leqslant u_p$.

(c) $\tau_p = \beta_p$ *if and only if, for each countable union of elements of* $\mathcal{P}$*, there exists an element of* $\mathcal{P}$ *containing this union.*

(d) $\beta_p = u_p$ *if and only if* $X \in \mathcal{P}$.

Proof. It is enough to replace the symbols $s(E)$, $\tau_{\mathcal{P}_m}$, $\beta_{\mathcal{P}_m}$, τ_u , $\beta_{\mathcal{P}}$ and $\tau_{\mathcal{P}}$ by $\{p\}$ $\tau_p(\mathcal{P}_m)$, $\beta_p(\mathcal{P}_m)$, u_p , β_p and τ_p respectively in the proof of 2.1.4, where $m = 1, 2$. □

We can summarize the different comparisons of the topologies $\beta_{\mathcal{P}}$ and β_p with the other topologies defined previously, regarding the relation $\leqslant$, as follows:

$$pw = \tau_{A(X)} \leqslant \tau_{\mathcal{P}} \leqslant \beta_{\mathcal{P}} \leqslant \beta_0 \leqslant \tau_u \qquad \tau_p(A(X)) \leqslant \tau_p \leqslant \beta_p \leqslant \beta_p(K(X)) \leqslant u_p$$

$$pw = \tau_{A(X)} \leqslant \delta \leqslant \beta_{\mathcal{P}} \leqslant \beta_0 \leqslant \tau_u \qquad \tau_p(A(X)) \leqslant \beta_p(A(X)) \leqslant \beta_p \leqslant \beta_p(K(X)) \leqslant u_p$$

Corollary 2.1.6. *The following statements are equivalent:*

(a) $X \in \mathcal{P}$.

(b) $\beta_p = u_p$*, for each* $p \in s(E)$.

(c) $\tau_p = \beta_p = u_p$*, for each* $p \in s(E)$.

(d) There exists some $p \in s(E)$*, for which,* $\tau_p = \beta_p = u_p$.

(e) There exists some $p \in s(E)$*, for which,* $\beta_p = u_p$.

(f) $\beta_{\mathcal{P}} = \tau_u$.

(g) $\tau_{\mathcal{P}} = \beta_{\mathcal{P}} = \tau_u$.

Proof. By 2.1.5.d, it is immediate to prove that $(a) \Leftrightarrow (b)$ and $(e) \Rightarrow (a)$. By the equivalence $(a) \Leftrightarrow (b)$ and by 2.1.5.c it is clear that $(b) \Leftrightarrow (c)$, $(c) \Rightarrow (d)$ and $(d) \Rightarrow (e)$.

Due to 2.1.4.d we have $(a) \Leftrightarrow (f)$. The last equivalence $(f) \Leftrightarrow (g)$ follows from the previous one and 2.1.4.c. □

We can make a small improvement of the previous corollary. The remark 2.1.3, alongside 2.1.4 and 2.1.6 results in the following corollary:

Corollary 2.1.7. *The following statements are equivalent:*

(a) $X \in \mathcal{P}$.

(b) $\beta_p = \beta_p(K(X)) = u_p$, *for each* $p \in s(E)$.

(c) $\tau_p = \tau_p(K(X)) = \beta_p = \beta_p(K(X)) = u_p$, *for each* $p \in s(E)$.

(d) There exists some $p \in s(E)$ *for which* $\tau_p = \tau_p(K(X)) = \beta_p = \beta_p(K(X)) = u_p$.

(e) There exists some $p \in s(E)$ *for which* $\beta_p = \beta_p(K(X)) = u_p$.

(f) $\beta_{\mathcal{P}} = \beta_0 = \tau_u$.

(g) $\tau_{\mathcal{P}} = \tau_{K(X)} = \beta_{\mathcal{P}} = \beta_0 = \tau_u$.

Corollary 2.1.8. *The following statements are equivalent:*

(a) Each countable union of elements of $\mathcal{P}$ *is contained in some element of* $\mathcal{P}$.

(b) $\tau_p = \beta_p$ *for each* $p \in s(E)$,

(c) There exists some $p \in s(E)$ *for which* $\tau_p = \beta_p$

(d) $\tau_{\mathcal{P}} = \beta_{\mathcal{P}}$

Proof. By 2.1.5.c, it is immediate to prove that $(a) \Leftrightarrow (b)$ and $(c) \Rightarrow (a)$, while $(b) \Rightarrow (c)$ is direct. Furthermore, due to 2.1.4.c we have $(a) \Leftrightarrow (d)$. □

Proposition 2.1.9. *If each countable union of elements of* $\mathcal{P}$ *is contained in some element of* $\mathcal{P}$, *then* X *is a pseudo-compact space.*

Proof. By contradiction, suppose there exists a continuous function f of X in $\mathbb{R}$ which is not bounded; since for each $n \in \mathbb{N}$, we can choose $x_n \in X$ such that $|f(x_n)| \geqslant n$. For each $n \in \mathbb{N}$, choose K_n in $\mathcal{P}$ such that $x_n \in K_n$. By hypothesis, there exists $K \in \mathcal{P}$ for which $K_n \subset K$ for all $n \in \mathbb{N}$. Thus, $x_n \in K$ for all $n \in \mathbb{N}$. On the other hand, by the continuity of f in K, there exists $M > 0$ such that $|f(x)| < M$ for each $x \in K$, which is a contradiction. □

2.2. $\beta_{\mathcal{P}}$-bounded sets in $C_b(X,E)$ and $B(X,E)$.

Proposition 2.2.1. *Let B be a subset of $C_b(X,E)$. The set B is τ_u-bounded if and only if it is $\beta_{\mathcal{P}}$-bounded.*

Proof. If B is τ_u-bounded, then B is $\beta_{\mathcal{P}}$-bounded since $\beta_{\mathcal{P}} \leqslant \tau_u$ (2.1.4.b). Conversely, suppose that B is not τ_u-bounded, then for some $p \in s(E)$ there exist a sequence (x_n) in X and another sequence (f_n) in B such that $p(f_n(x_n)) \geqslant n^3$. By 1.2.7 $v = \sum_{n=1}^{\infty} n^{-2}\mathscr{X}_{\{x_n\}} \in V_{\mathcal{P}}$. Then we have $\|f_n\|_{p,v} \geqslant v(x_n)p(f_n(x_n)) \geqslant v(x_n)n^3 = n$ for each $n \in \mathbb{N}$. Which proves that B is not $\beta_{\mathcal{P}}$-bounded. □

Analogously we have:

Proposition 2.2.2. *Let B be a subset of $C_b(X,E)$. The set B is u_p-bounded if and only if it is β_p-bounded.*

Proof. The proof is obtained by replacing the symbols $s(E)$, τ_u and $\beta_{\mathcal{P}}$ by $\{p\}$, u_p and β_p respectively in the proof of the previous proposition. □

Proposition 2.2.3. *In the space $C_b(X,E)$ the topologies $\beta_{\mathcal{P}}$ and $\tau_{\mathcal{P}}$ coincide over τ_u-bounded sets.*

Proof. Let B be a subset τ_u-bounded of $C_b(X,E)$. Since $\tau_{\mathcal{P}} \leqslant \beta_{\mathcal{P}}$, it is enough to prove that each basic neighborhood of 0 induced by $\beta_{\mathcal{P}}$ in B, contains a basic neighborhood of 0 induced by $\tau_{\mathcal{P}}$ in B. Considering $p \in s(E)$, $v \in V_{\mathcal{P}}$ and $a = \sup\{\|f\|_p : f \in B\}$, there exists a compact $K \in \mathcal{P}$ such that $\|v\|_{X\setminus K} \leqslant 1/(1+a)$. Therefore, for each $f \in B$ we have:

$$\|f\|_{p,v} = \text{máx}\left\{\sup_{x\in K} v(x)p(f(x)),\ \sup_{x\in X\setminus K} v(x)p(f(x))\right\} \leqslant \text{máx}\left\{\|v\| \sup_{x\in K} p(f(x)), 1\right\}$$

where the equality it is obtained from the characterization of the supremum. Therefore, if $f \in B$ satisfies the inequality $\sup\{p(f(x)) : x \in K\} \leqslant 1/(1+\|v\|)$, then $\|f\|_{p,v} \leqslant 1$. From this we obtain:

$$B \cap \left\{ f \in C_b(X,E) : \|f\|_{p,K} \leqslant \frac{1}{1+\|v\|} \right\} \subset B \cap \{f \in C_b(X,E) : \|f\|_{p,v} \leqslant 1\}.$$

Therefore, $\tau_{\mathcal{P}}$ and $\beta_{\mathcal{P}}$ coincide in B. □

Analogously we have:

Proposition 2.2.4. *In the space $C_b(X,E)$ the topologies β_p and τ_p coincide over u_p-bounded sets.*

Proof. The proof is obtained by replacing the symbols $s(E)$, τ_u ,$\tau_{\mathcal{P}}$ and $\beta_{\mathcal{P}}$ by $\{p\}$, u_p ,τ_p and β_p respectively in the proof of the previous proposition. □

2.3. The spaces E and $C_b(X)$ as subspaces of $(C_b(X,E),\beta_{\mathcal{P}})$.

In this section, using linear homeomorphisms, we will prove that the spaces E and $(C_b(X),\gamma_{\mathcal{P}})$ can be identified as $\beta_{\mathcal{P}}$-closed subspaces of $C_b(X,E)$.

Definition 2.3.1. *Given a vector space V, a projection on V is a linear map $T : V \to V$ such that $T = T^2$.*

Remark 2.3.2. *If T is a projection on V, then, $V = \operatorname{Ker}(T) \oplus \operatorname{Im}(T)$. Conversely, if N and M are subsets of V such that $V = N \oplus M$, then there exists a projection T on V such that $\operatorname{Ker}(T) = N$, $\operatorname{Im}(T) = M$ and $T|_M = Id_M$. Furthermore, $P := Id_V - T$ is also a projection on V such that $\ker(P) = M$, $\operatorname{Im}(P) = N$ and $P|_N = Id_N$.*

Definition 2.3.3. *If N and M are subsets of V such that $V = N \oplus M$, then we will say they are **complemented subspaces of V**.*

*Let T be the projection such that $T|_N = Id_N$. When V is equipped with a Hausdorff vector topology and T is continuous, we will say that N is a **topologically complemented subspace of V**.*

Using the previous observation, we have the following lemma.

Lemma 2.3.4. *All topologically complemented subspace of a Hausdorff topological vector space is a closed subspace.*

Proposition 2.3.5.

(a) For a fix $x_0 \in X$, the map $T_1 : (C_b(X,E),\beta_{\mathcal{P}}) \to E$ defined by $T_1(f) = f(x_0)$, is linear and continuous.

(b) For a fix $f_0 \in C_b(X)$, $f_0 \neq 0$, the map $T_2 : E \to (C_b(X,E),\beta_{\mathcal{P}})$ defined by $T_2(e) = f_0 \otimes e$, is linear, continuous and injective. It is also an isomorphism between E and $T_2(E)$, equipped with the subspace topology induced by $\beta_{\mathcal{P}}$.

(c) The map $T_2 \circ T_1 : (C_b(X,E),\beta_{\mathcal{P}}) \to (C_b(X,E),\beta_{\mathcal{P}})$, defined by $T_2 \circ T_1(f) = f_0 \otimes f(x_0)$, is linear and continuous. If $f_0(x_0) = 1$, then:

- $T_1 \circ T_2 = Id_E$.
- $(T_2 \circ T_1)^2 = (T_2 \circ T_1)$.
- $T_2 \circ T_1(C_b(X,E)) = T_2(E)$.
- $T_2(E) \oplus \operatorname{Ker}(T_2 \circ T_1) = C_b(X,E)$.

Moreover E is isomorphic to a topologically complemented subspace of $C_b(X,E)$.

Proof. (a) The linearity of T_1 is clear. For the continuity, consider $p \in s(E)$ and $f \in C_b(X,E)$. By the fact that $p(T_1(f)) = p(f(x_0))$, T_1 is (pw,τ_E)-continuous and then $(\beta_{\mathcal{P}},\tau_E)$-continuous.

(b) The linearity of T_2 is clear. To prove that T_2 is continuous, note that for a fix $p \in s(E)$, we have:

$$\|T_2(e)\|_p = \|f_0 \otimes e\|_p = \sup_{x\in X} |f_0(x)|p(e) = \|f_0\|p(e)$$

that is T_2 is (τ_E,τ_u)-continuous, which implies that it is $(\tau_E,\beta_{\mathcal{P}})$-continuous. Now, consider $e \in E$ such that $T_2(e) = 0$. Since $f_0(x) \neq 0$ for some $x \in X$, we have that

$f_0(x)e = 0$ and then $e = 0$. This prove that T_2 is injective note that $T_2^{-1} : T_2(E) \to E$ defined by $T_2^{-1}(f_0 \otimes e) := e$, is also injective.

To prove that T_2 is an isomorphism from E into $T_2(E)$, equipped with the topology induced by $\beta_{\mathcal{P}}$, it is enough to prove that T_2^{-1} is continuous regarding this topology.

For this, we choose $y \in X$, such that $f_0(y) \neq 0$. For each $p \in s(E)$ and $e \in E$, we have:

$$p(T_2^{-1}(f_0 \otimes e)) = p(e) = \frac{p(f_0(y)e)}{|f_0(y)|}$$

which proves the continuity of T_2^{-1} respect to the induced *pw* topology induced $\beta_{\mathcal{P}}$ topology.

(c) Since the maps T_1 and T_2 are linear and continuous, so $T_2 \circ T_1$ is. Suppose $f_0(x_0) = 1$ and consider $e \in E$.

$$(T_1 \circ T_2)(e) = T_1(f_0 \otimes e) = f_0(x_0)e = e$$

therefore, $(T_1 \circ T_2) = Id_E$. Using this fact, we get

$$(T_2 \circ T_1)^2 = (T_2 \circ T_1) \circ (T_2 \circ T_1) = T_2 \circ (T_1 \circ T_2) \circ T_1 = T_2 \circ Id_E \circ T_1 = T_2 \circ T_1.$$

And since $T_2(E) \subset C_b(X,E)$, $T_2(E) = (T_2 \circ T_1 \circ T_2)(E) \subset T_2 \circ T_1(C_b(X,E))$.

Also, $T_1(C_b(X,E)) \subset E$, implies, $T_2 \circ T_1(C_b(X,E)) \subset T_2(E)$. From this, it is concluded that $T_2 \circ T_1(C_b(X,E)) = T_2(E)$. On the other hand, since the map $T_2 \circ T_1$ is a projection on $C_b(X,E)$, we finitely get

$$C_b(X,E) = \mathrm{Im}(T_2 \circ T_1) \oplus \mathrm{Ker}(T_2 \circ T_1) = T_2(E) \oplus \mathrm{Ker}(T_2 \circ T_1).$$

□

Proposition 2.3.6.

(a) For $e_0' \in E'$, with a fix $e_0' \neq 0$, the map $S_1 : (C_b(X,E),\beta_{\mathcal{P}}) \to (C_b(X),\gamma_{\mathcal{P}})$ defined by $S_1(f) = e_0' \circ f$, is linear and continuous.

(b) For a fix $e_0 \in E$, $e_0 \neq 0$, the map $S_2 : (C_b(X),\gamma_{\mathcal{P}}) \to (C_b(X,E),\beta_{\mathcal{P}})$ defined by $S_2(f) = f \otimes e_0$, is linear, continuous and injective. It is also an isomorphism from $(C_b(X),\gamma_{\mathcal{P}})$ onto $s_2(C_b(X))$, equipped with the topology induced by $\beta_{\mathcal{P}}$.

(c) *The map* $S_2 \circ S_1 : (C_b(X,E),\beta_{\mathcal{P}}) \to (C_b(X,E),\beta_{\mathcal{P}})$ *defined by* $S_2 \circ S_1(f) = (e_0' \circ f) \otimes e_0$, *is linear and continuous. If* $e_0'(e_0) = 1$, *then:*

- $S_1 \circ S_2 = Id_{C_b(X)}$.
- $(S_2 \circ S_1)^2 = (S_2 \circ S_1)$.
- $S_2 \circ S_1(C_b(X,E)) = S_2(C_b(X))$.
- $S_2(C_b(X)) \oplus \mathrm{Ker}(S_2 \circ S_1) = C_b(X,E)$.

In particular, $C_b(X)$ *is isomorphic to a topologically complemented subspace of* $C_b(X,E)$.

Proof. (a) The linearity of S_1 is clear. For the continuity, notice that $|e_0'(\cdot)|$ is a continuous seminorm in E; hence, there exist $p \in s(E)$ and $r > 0$ such that $|e_0'(e)| \leqslant rp(e)$, for all $e \in E$. Therefore, given $v \in V_{\mathcal{P}}$, we have:

$$\|S_1(f)\|_v = \|e_0' \circ f\|_v = \sup_{x\in X} v(x)|e_0'(f(x))| \leqslant \sup_{x\in X} rv(x)p(f(x)) = r\|f\|_{p,v}$$

where $f \in C_b(X,E)$. This prove that S_1 is $(\beta_{\mathcal{P}},\gamma_{\mathcal{P}})$-continuous.

(b) The linearity of S_2 is clear. Let $p \in s(E)$ and $v \in V_{\mathcal{P}}$. Since for each $f \in C_b(X)$, $\|S_2(f)\|_{p,v} = \|f \otimes e_0\|_{p,v} = \|f\|_v p(e_0)$, we have that S_2 is $(\gamma_{\mathcal{P}},\beta_{\mathcal{P}})$-continuous. We claim that S_2 is injective; in fact, if $S_2(f) = 0$, then $f(x)e_0 = 0$ for all $x \in X$ which implies that $f = 0$. The inverse linear operator $S_2^{-1} : S_2(C_b(X)) \to C_b(X)$ in such that $S_2^{-1}(f \otimes e_0) := f$.

To prove that S_2 is an isomorphism from $(C_b(X),\gamma_{\mathcal{P}})$ onto $S_2(C_b(X))$, equipped with the topology induced by $\beta_{\mathcal{P}}$, it is enough to prove that S_2^{-1} is continuous.

For this, we choose $p \in s(E)$, such that $p(e_0) \neq 0$. For each $v \in V_{\mathcal{P}}$ and $f \in C_b(X)$, we have:

$$\|S_2^{-1}(f)\|_v = \|f\|_v = \frac{\|f\|_v p(e_0)}{p(e_0)} = \frac{\|f \otimes e_0\|_{p,v}}{p(e_0)}$$

which proves the continuity of S_2^{-1}.

(c) Since the maps S_1 and S_2 are linear and continuous, so $S_2\circ S_1$ is. Suppose $e_0'(e_0)=1$ and consider $g\in C_b(X)$.

$$(S_1\circ S_2)(g)=S_1(g\otimes e_0)=g\cdot e_0'(e_0)=g$$

Therefore, $(S_1\circ S_2)=Id_{C_b(X)}$. Consequently,

$$(S_2\circ S_1)^2=S_2\circ S_1\circ S_2\circ S_1=S_2\circ Id_E\circ S_1=S_2\circ S_1.$$

Then, $S_2(C_b(X))=(S_2\circ S_1\circ S_2)(C_b(X))\subset S_2\circ S_1(C_b(X,E))$, since $S_2(C_b(X))\subset C_b(X,E)$. Also, $S_1(C_b(X,E))\subset C_b(X)$, therefore, $S_2\circ S_1(C_b(X,E))\subset S_2(C_b(X))$.

From this, it is concluded that $S_2\circ S_1(C_b(X,E))=S_2(C_b(X))$. Then, since the map $S_2\circ S_1$ is a projection on $C_b(X,E)$, we have

$$C_b(X,E)=\mathrm{Im}(S_2\circ S_1)\oplus\mathrm{Ker}(S_2\circ S_1)=S_2(C_b(X))\oplus\mathrm{Ker}(S_2\circ S_1).$$

□

The following corollaries are very useful tools for future results and are an immediate consequence of the previous propositions.

Corollary 2.3.7. *The spaces E and $C_b(X)$ are subspaces $\beta_{\mathcal{P}}$-closed of $C_b(X,E)$.*

Proof. It is immediately from 2.3.5, 2.3.6 and 2.3.4. □

Corollary 2.3.8. *If the space $(C_b(X,E),\beta_{\mathcal{P}})$ is complete (respectively quasi-complete), then, the spaces E and $(C_b(X),\gamma_{\mathcal{P}})$ are completes (respectively quasi-completes).*

Proof. The proof is direct from 2.3.7, since E and $(C_b(X),\gamma_{\mathcal{P}})$ can be identified as closed subspaces of $(C_b(X,E),\beta_{\mathcal{P}})$, by linear homeomorphisms. □

Corollary 2.3.9. *If the space $(C_b(X,E),\beta_{\mathcal{P}})$ is separable, then, the spaces E and $(C_b(X),\gamma_{\mathcal{P}})$ are separable.*

Proof. Consider the maps T_1, T_2, S_1 and S_2 defined in 2.3.5 and 2.3.6.

Let D be a countable subset and $\beta_{\mathcal{P}}$-dense of $C_b(X, E)$. By 2.3.5 we have:

$$T_2(E) = T_2 \circ T_1(C_b(X, E)) = T_2 \circ T_1(\overline{D}) \subset \overline{T_2 \circ T_1(D)} \subset \overline{T_2 \circ T_1(C_b(X, E))} = \overline{T_2(E)} \ .$$

Thereby, by 2.3.7, $\overline{T_2 \circ T_1(D)} = \overline{T_2(E)} = T_2(E)$, proving that $T_2(E)$ is $\beta_{\mathcal{P}}$-separable. Then, by the isomorphism $T_2 : E \to T_2(E)$, it is concluded that E is $\tau_{_E}$-separable.

Analogously, applying 2.3.6, we have:

$$\begin{aligned} S_2(C_b(X)) = S_2 \circ S_1(C_b(X, E)) = S_2 \circ S_1(\overline{D}) \subset \overline{S_2 \circ S_1(D)} &\subset \overline{S_2 \circ S_1(C_b(X, E))} \\ &= \overline{S_2(C_b(X))} \ . \end{aligned}$$

Thereby, by 2.3.7, $\overline{S_2 \circ S_1(D)} = \overline{S_2(C_b(X))} = S_2(C_b(X))$, proving that $S_2(C_b(X))$ is $\beta_{\mathcal{P}}$-separable. Then, by the isomorphism $S_2 : C_b(X) \to S_2(C_b(X))$, it is concluded that $C_b(X)$ is $\gamma_{\mathcal{P}}$-separable. ☐

2.4. The $\beta_{\mathcal{P}}$-density of $C_b(X) \otimes E$ in $C_b(X, E)$.

Before stating the main result of this section, we will prove a few preliminary lemmas.

Lemma 2.4.1. *Let F and K be disjoint subspaces of a completely regular Hausdorff space X, such that F is closed and K is compact. There exist open neighborhoods U_0 and U_1 of F and K respectively, that are disjoint with each other, and a continuous function $f : X \to \mathbb{R}$ such that $0 \leqslant f \leqslant \mathscr{X}_X$, $f(U_0) = \{0\}$ and $f(U_1) = \{1\}$.*

Proof. We can identify X with a topological subspace of its compactification of Stone-Čech βX, and therefore, we can identify K with a compact subspace of βX. Also, there exists a subset C of βX closed, such that $F = X \cap C$. Since K and F are disjoint, we have that K and C also are. Since βX is a normal space, by Urysohn characterization there exists a continuous function $g : \beta X \to \mathbb{R}$ such that $0 \leqslant g \leqslant \mathscr{X}_{\beta X}$, $g(C) = \{0\}$ and $g(K) = \{1\}$. If we consider the function $h := g|_X$ then, $0 \leqslant h \leqslant \mathscr{X}_X$, $h(F) = \{0\}$ and $h(K) = \{1\}$. The sets $U_0 = \{x \in X : h(x) < 1/3\}$ and $U_1 = \{x \in X : h(x) > 2/3\}$ are open neighborhoods of F and K respectively. We take a continuous function

$t : [0,1] \to [0,1]$ such that $t(\left[0, \frac{1}{3}\right]) = \{0\}$ and $t(\left[\frac{2}{3}, 1\right]) = \{1\}$. We finish the proof by defining the function $f := t \circ h$, which satisfies all of the requirements. $\square$

Lemma 2.4.2. *If K is a compact subspace of X and $\{U_i : 1 \leqslant i \leqslant n\}$ is a finite collection of open subsets of X, such that $K \subset \bigcup_{i=1}^{n} U_i$ and $K \not\subset \bigcup\{U_i : i \in \{1, \dots, n\} - \{j\}\}$ for each $j = 1, \dots, n$, then there exists an open neighborhood U of K and a collection of functions $\{f_i : 1 \leqslant i \leqslant n\} \subset C_b(X)$, such that $0 \leqslant \sum_{i=1}^{n} f_i \leqslant \mathscr{X}_X$, $\sum_{i=1}^{n} f_i(x) = 1$ for all $x \in K$ and for each $i \in \{1, \dots, n\}$, we have:* $\operatorname{supp} f_i \subset U_i$.

Proof. Defining the set

$$K_1 := K \backslash \bigcup_{i=2}^{n} U_i$$

which is compact and contained in U_1. By lemma 2.4.1, there exist disjoint open neighborhoods V_0 and V_1 of $X \backslash U_1$ and K_1 respectively, and also there exists a function $g_1 \in C_b(X)$ such that $0 \leqslant g_1 \leqslant \mathscr{X}_X$, $g_1(V_0) = \{0\}$ and $g_1(V_1) = \{1\}$. Then $\operatorname{supp} g_1 \subset U_1$ and $V_1 \subset U_1$.

Defining the set

$$K_2 := \left(K \backslash \bigcup_{i=3}^{n} U_i \right) \backslash V_1 .$$

which is a compact set contained in U_2. In fact, if $x \in K_2 \backslash U_2$, then $x \in K \backslash \left(\bigcup_{i=2}^{n} U_i\right) = K_1 \subset V_1$, reaching a contradiction, since $x \notin V_1$. Then K_2 is a disjoint compact of the closed set $X \backslash U_2$. Applying the lemma 2.4.1 there exists an open neighborhood V_2 of K_2 and a function $g_2 \in C_b(X)$ such that $V_2 \subset U_2$, $0 \leqslant g_2 \leqslant \mathscr{X}_X$, $g_2(V_2) = \{1\}$ and $\operatorname{supp} g_2 \subset U_2$.

Recursively, for $\ell \in \{3, \dots, n-1\}$ we define

$$K_\ell := \left(K \backslash \bigcup_{i=\ell+1}^{n} U_i \right) \backslash \left(\bigcup_{i=1}^{\ell-1} V_i \right) ,$$

which is a compact set contained in U_ℓ. Also, there exists an open neighborhood V_ℓ of K_ℓ and a function $g_\ell \in C_b(X)$ such that $V_\ell \subset U_\ell$, $0 \leqslant g_\ell \leqslant \mathscr{X}_X$, $g_\ell(V_\ell) = \{1\}$ and $\operatorname{supp} g_\ell \subset U_\ell$.

Finnaly, defining the set

$$K_n := K \setminus \left(\bigcup_{i=1}^{n-1} V_i \right)$$

which is a compact set contained in U_n. In fact, if $x \in K_n \setminus U_n$, then $x \in (K \setminus U_n) \setminus \left(\bigcup_{i=1}^{n-2} V_i \right) = K_{n-1} \subset V_{n-1}$, reaching a contradiction, since $K_n \cap V_{n-1} = \emptyset$.

Applying the lemma 2.4.1, we obtain the existence of a function $g_n \in C_b(X)$ such that $g_n(K_n) = \{1\}$ and $\operatorname{supp} g_n \subset U_n$. Since $K \subset K_n \cup \bigcup_{i=1}^{n-1} V_i$, the function $\sum_{i=1}^{n} g_i$ is greater or equal than 1 at each point of K. The function $g := \text{mín}\{1, \sum_{i=1}^{n} g_i\}$ belongs to $C_b(X)$ and it satisfies $0 \leqslant g \leqslant \mathscr{X}_X$ and $g(K) = \{1\}$. Then, the set

$$U = \{x \in X : g(x) > 1/4\} = \{x \in X : \sum_{i=1}^{n} g_i(x) > 1/4\}$$

is an open neighborhood of K. Taking a continuous function $t : [0, 1] \to [0, 1]$ such that, $t\left(\left[0, \frac{1}{3}\right]\right) = \{0\}$ and $t\left(\left[\frac{2}{3}, 1\right]\right) = \{1\}$. Thus, the function $h := t \circ g$ belongs to $C_b(X)$ and satisfies: $0 \leqslant h \leqslant \mathscr{X}_X$, $h(K) = \{1\}$ and $h(\{x \in X : g(x) < 1/3\}) = \{0\}$. Then $\operatorname{supp} h \subset U$, in fact, if $y \notin U$, then $U := \{x \in X : g(x) < 1/3\}$ is an open neighborhood of y, for which $h(U) = \{0\}$ and from this $U \cap \{x \in X : h(x) \neq 0\} = \emptyset$. Therefore $y \notin \operatorname{supp} h$. For each $i = 1, \ldots, n$, we define the function f_i by

$$f_i(x) = \begin{cases} \dfrac{h(x) g_i(x)}{\sum_{i=1}^{n} g_i(x)} & , \text{ si } x \in U \\ 0 & , \text{ si } x \in X \setminus U \ . \end{cases}$$

To finalize the proof, it remains to prove that $f_i \in C_b(X)$ for each $i \in \{1, \ldots, n\}$.

Let $i \in \{1, \ldots, n\}$. The function f_i is bounded, since $f_i \leqslant h$. Since $\frac{h \cdot g_i}{\sum_{i=1}^{n} g_i}\big|_U : U \to \mathbb{R}$ is a continuous function and U is an open set in X, we have that f_i is continuous at each point of U. If $y \in X \setminus U$, then $h(y) = 0$. Given $\varepsilon > 0$, $U := \{x \in X : h(x) < \varepsilon\}$ is a neighborhood of y. Since $|f_i(x)| \leqslant h(x) < \varepsilon$ for any $x \in U$, it is concluded that f_i is continuous in y. Thereby $f_i \in C_b(X)$. □

Theorem 2.4.3. *The space $C_b(X) \otimes E$ is $\beta_{\mathcal{P}}$-dense in $C_b(X, E)$.*

Proof. Let $f \in C_b(X,E)$, $p \in s(E)$, $v \in V_{\mathcal{P}}$ and $\varepsilon > 0$. There exists $K \in \mathcal{P}$ such that: $\|v\|_{X\setminus K} \leqslant \varepsilon/(2\|f\|_p + 1)$. Notice that

$$f(K) \subset \bigcup_{x\in K} \left\{ e \in E : p(e - f(x)) < \frac{\varepsilon}{\|v\| + 1} \right\} .$$

By the continuity of f, we have that $f(K)$ is a compact subset of E. Then, there exist $x_1, \ldots, x_n$ elements of K, such that

$$f(K) \subset \bigcup_{i=1}^{n} \left\{ e \in E : p(e - f(x_i)) < \frac{\varepsilon}{\|v\| + 1} \right\} .$$

Defining

$$U_i = f^{-1}\left(\left\{ e \in E : p(e - f(x_i)) < \frac{\varepsilon}{\|v\| + 1} \right\} \right)$$

we obtain the open collection $\{U_i : 1 \leqslant i \leqslant n\}$, which is a finite covering of K. By 2.4.2, for each $i = 1, \ldots, n$, there exist continuous functions $g_i : X \to [0,1]$ such that: $\mathrm{supp}(g_i) \subset U_i$, $0 \leqslant \sum_{i=1}^{n} g_i \leqslant \mathscr{X}_X$ and $\sum_{i=1}^{n} g_i(V) = \{1\}$, for some V open subset of X which contains K. Then:

$$\|f - \sum_{i=1}^{n} g_i \otimes f(x_i)\|_{p,v} = \sup_{x\in X} v(x) p \left(f(x) - \sum_{i=1}^{n} g_i(x) f(x_i) \right)$$

$$= \text{máx} \left\{ \sup_{x\in X\setminus K} v(x) p \left(f(x) - \sum_{i=1}^{n} g_i(x) f(x_i) \right), \sup_{x\in K} v(x) p \left(\sum_{i=1}^{n} g_i(x)(f(x) - f(x_i)) \right) \right\}$$

$$\leqslant \text{máx} \left\{ \|v\|_{X\setminus K} \cdot \sup_{x\in X\setminus K} p(f(x)) + \sum_{i=1}^{n} |g_i(x)| p(f(x_i)),\ \|v\| \sup_{x\in K} \sum_{i=1}^{n} |g_i(x)| p(f(x) - f(x_i)) \right\}$$

$$\leqslant \text{máx} \left\{ \frac{\varepsilon}{2\|f\|_p + 1} \cdot \left(\|f\|_p + \sup_{x\in X} \sum_{i=1}^{n} |g_i(x)| \|f\|_p \right), \|v\| \sup_{x\in K} \sum_{i=1}^{n} |g_i(x)| \frac{\varepsilon}{\|v\| + 1} \right\}$$

$$\leqslant \text{máx} \left\{ \frac{\varepsilon \cdot 2\|f\|_p}{2\|f\|_p + 1}, \frac{\|v\|\varepsilon}{\|v\| + 1} \right\} \leqslant \varepsilon .$$

□

2.5. Topological properties of $(C_b(X,E),\beta_{\mathcal{P}})$.

In this section we study the necessary and sufficient conditions to make the space $(C_b(X,E),\beta_{\mathcal{P}})$ be bornological, barreled, quasibarreled among others. In the appendix C are proved the different equivalent definitions of these spaces that will be used in this document.

Proposition 2.5.1. *If L is a topologically complemented subspace of E and E is ultrabornological, then L is ultrabornological.*

Proof. Let $T : E \to L$ be a continuous projection such that $T|_L = Id_L$. Suppose that E is a ultrabornological space. It is enough to prove that a seminorm p in L is continuous, if it is bounded in each absolutely convex and compact subset of L. This by C.3.8. Notice that $p \circ T$ is a seminorm in E which coincide with p in L, and it is continuous in E since this is bounded in each absolutely convex and compact subset of L. Then $p = p \circ T|_L$ is continuous. □

Remark 2.5.2. *If L is a topologically complemented subspace of E and E is bornological (respectively barreled)(resp. quasibarreled), then L is bornological (resp. barreled)(resp. quasibarreled).*

To prove this, we use similar arguments to those used in 2.5.1. In fact, if E is a space:

(a) bornological,

(b) barreled,

(c) quasibarreled,

enough replacing in the proof of 2.5.1 the condition "a seminorm p in L is continuous, if it is bounded at each absolutely convex and compact subset of L" by: "a seminorm p in L is continuous, if:

(a) it is bounded at each bounded subset of L",

(b) it is semi-continuous inferiorly in L'',

(c) it is semi-continuous inferiorly in L and bounded at each bounded subset of L'',

respectively, and instead of using the definition C.3.8, is used the definition:

(a) C.3.2,

(b) C.1.10,

(c) C.1.7,

respectively.

Proposition 2.5.3. *If L is a topologically complemented subspace of E and E is a (DF)-space, then L is a (DF)-space.*

Proof. Let $T : E \to L$ be a continuous projection such that $T|_L = Id_L$. Suppose that E is a (DF)-space (using the definition C.2.6). Note that if $\{B_n : n \in \mathbb{N}\}$ is a fundamental system of bounded in E, then $\{L \cap B_n : n \in \mathbb{N}\}$ is a fundamental system of bounded in L. Furthermore, let $(q_n)_{n\in\mathbb{N}}$ be a sequence of continuous seminorms in L such that $q = \sup_{n\in\mathbb{N}} q_n$ is a seminorm in L bounded in the bounded subsets of L. Then, $(q_n \circ T)_{n\in\mathbb{N}}$ is a sequence of continuous seminorms in L, such that $q \circ T = \sup_{n\in\mathbb{N}} q_n \circ T$ is a seminorm in E bounded in the bounded subsets of E. Then, $q \circ T$ is a continuous seminorm in E and from this $q \circ T|_L = q$ is a continuous seminorm in L. □

Introducing the following definition, from [22, page 86].

Definition 2.5.4. *A sequence $\underline{e} = (e_n)_{n\in\mathbb{N}}$ in E is **absolutely summable** if for each $p \in s(E)$, we have:*

$$\pi_p(\underline{e}) := \sum_{n=1}^{\infty} p(e_n) < \infty.$$

The space of all sequences in E absolutely summable will be denoted by $\ell^1(E)$ and will be equipped with the topology $\pi := \sigma(\ell^1(E), \{\pi_p : p \in s(E)\})$.

We will say that ***E has the property (B)*** *if for each* $\mathscr{B}\subset\ell^1(E)$, *π-bounded, there exists* $B\subset E$, $\tau_{_E}$*-bounded absolutely convex, such that:*

$$\sum_{n=1}^{\infty} p_B(e_n)\leqslant 1 \quad \forall (e_n)_n\in\mathscr{B}$$

where $p_B:\operatorname{span}(B)\to\mathbb{R}$ *is the Minkoski functional of* B.

Proposition 2.5.5. *Suppose that* X *is not finite. The space* $(C_b(X,E),\beta_{\mathcal{P}})$ *is quasibarreled (respectively barreled) if and only if* $X\in\mathcal{P}$, E *is quasibarreled (respectively barreled) and* E'_b *has the property* (B).

Proof. If $(C_b(X,E),\beta_{\mathcal{P}})$ is quasibarreled (resp. barreled), then by 2.5.2 alongside 2.3.7, $(C_b(X),\gamma_{\mathcal{P}})$ and E are quasibarreled (resp. barreled). Since $(C_b(X),\gamma_{\mathcal{P}})$ and $(C_b(X),\tau_{\|\cdot\|})$ have the same bounded sets (this by 2.2.2), the set $\{f\in C_b(X):\|f\|\leqslant 1\}$ is a bornivorous barrel of $(C_b(X),\gamma_{\mathcal{P}})$. Therefore $\gamma_{\mathcal{P}}=\tau_{\|\cdot\|}$. Applying the corollary 2.1.6 it is concluded that $X\in\mathcal{P}$.

Then, by [22, iv.6.6,$(a)\Rightarrow(c)$] we have that E'_b has the property (B).

Now suppose that $X\in\mathcal{P}$, E is quasibarreled (resp. barreled) and E'_b has the property (B). In this case X is compact. Then, by [22, iv.6.6,$(c)\Rightarrow(b)$] (resp. by [22, iv.7.7,$(c)\Rightarrow(b)$]) the space $(C_b(X,E),\tau_{K(X)})$ (which in [22] is denoted by $C_c(K,E)$, where $K=X$) is quasibarreled (resp. barreled). Finally, by 2.1.7.g, it is concluded that $(C_b(X,E),\beta_{\mathcal{P}})$ is quasibarreled (resp. barreled). □

Corollary 2.5.6. *If* X *is not finite and the space* $(C_b(X,E),\beta_{\mathcal{P}})$ *is bornological (respectively ultrabornological), then* $X\in\mathcal{P}$, E *is bornological (respectively ultrabornological) and* E'_b *has the property* (B).

Proof. By 2.5.2 (resp. 2.5.1) alongside 2.3.7, E is bornological (resp. ultrabornological). If $(C_b(X,E),\beta_{\mathcal{P}})$ is bornological or ultrabornological, then is quasibarreled and since X is not finite, by 2.5.5, it is obtained the result. □

Proposition 2.5.7. *The space* $(C_b(X,E),\beta_{\mathcal{P}})$ *is a (DF)-space if and only if* E *is a (DF)-space and all countable union of elements of* $\mathcal{P}$ *are contained in some element of* $\mathcal{P}$.

Proof. Suppose that $(C_b(X,E),\beta_{\mathcal{P}})$ is a (DF)-space. By 2.5.3 alongside 2.3.7, $(C_b(X),\gamma_{\mathcal{P}})$ and E are (DF)-spaces. By [16, 9.5.2] we have that $\gamma_{\mathcal{P}} = \tau_{\mathcal{P}}$ and from this all countable union of elements of $\mathcal{P}$ are contained in some element of $\mathcal{P}$ (this by 2.1.4.c).

Now suppose that E is a (DF)-space and all countable union of elements of $\mathcal{P}$ is contained in some element of $\mathcal{P}$. Once again by 2.1.4.c, we have $\gamma_{\mathcal{P}} = \tau_{\mathcal{P}}$ and applying [16, 9.5.2] alongside [22, iv.9.2,(a)] it is obtained that $(C_b(X,E),\beta_{\mathcal{P}})$ is a (DF)-space. □

Proposition 2.5.8. *The space $(C_b(X,E),\beta_{\mathcal{P}})$ is a (gDF)-space if and only if E is a (gDF)-space.*

Proof. The proof of this result is widely developed in [23]. □

In the following, we will study the separability of the space $(C_b(X,E),\beta_{\mathcal{P}})$. We will show the following definition, from [31, page 8].

Definition 2.5.9. *A topological space $\mathcal{X}$ is called **separably submetrizable** if the topology in $\mathcal{X}$ is finer than some metrizable and separable topology in $\mathcal{X}$.*

Equivalently, $\mathcal{X}$ is separably submetrizable, if and only if, there exists a continuous and injective function in $\mathcal{X}$ with values in a metric and separable space.

The following result was extracted from [25, Theorem 2.1, page 509].

Proposition 2.5.10. *The space $(C_b(X),\gamma_0)$ is separable if and only if X is separably submetrizable.*

Proposition 2.5.11. *The following statements are equivalent:*

(a) X is separably submetrizable and E is separable.

(b) $(C_b(X,E),\beta_{\mathcal{P}})$ is separable.

(c) $(C_b(X,E),\delta)$ is separable.

Proof. $(a) \Rightarrow (b)$: By the previous proposition, the space $(C_b(X),\gamma_0)$ is separable and, from this, $(C_b(X),\gamma_{\mathcal{P}})$ is also separable, since $\gamma_{\mathcal{P}} \leqslant \gamma_0$. Since the product of two separable spaces is separable, we have $C_b(X)\times E$ with the product topology is separable. Considering the linear map $L : C_b(X) \times E \to C_b(X) \otimes E$, defined by $L(f,e) := f \otimes e$. If the space $C_b(X) \otimes E$ is equipped with the induced subspace topology by $\beta_{\mathcal{P}}$, then L is continuous. In fact, let $(f_\alpha, e_\alpha)_{\alpha\in\Lambda}$ be a net in $C_b(X) \times E$ convergent to $(0,0)$. Then $(f_\alpha)_{\alpha\in\Lambda}$ is a net $\gamma_{\mathcal{P}}$-converged to 0 and $(e_\alpha)_{\alpha\in\Lambda}$ a net to E convergent to 0. Let $v \in V_{\mathcal{P}}$ and $p \in s(E)$. For $\alpha \in \Lambda$, we have:

$$\begin{aligned} \|f_\alpha \otimes e_\alpha\|_{v,p} &= \sup_{x\in X} v(x)p(f_\alpha(x)e_\alpha) \\ &= \sup_{x\in X} v(x)|f_\alpha(x)|p(e_\alpha) \\ &= \|f_\alpha\|_v \cdot p(e_\alpha) \end{aligned}$$

concluding that $(f_\alpha \otimes e_\alpha)_{\alpha\in\Lambda}$ is $\beta_{\mathcal{P}}$-converged to $(0,0)$ in $C_b(X) \times E$. Therefore L is continuous. Thus, we can affirm that $L(C_b(X) \times E)$ is separable and since $\operatorname{span} L(C_b(X)\times E) = C_b(X)\otimes E$, it follows that $C_b(X)\otimes E$ is separable. Since $C_b(X)\otimes E$ is $\beta_{\mathcal{P}}$-dense in $C_b(X,E)$ (2.4.3), we have that (b) is satisfied.

$(b) \Rightarrow (c)$: It is clear.

$(c) \Rightarrow (a)$: By the result 2.3.9, we know that E and $(C_b(X),\gamma_{A(X)})$ are separable spaces. Let $D := \{f_m : m \in \mathbb{N}\}$ be a subset $\gamma_{A(X)}$-dense in $C_b(X)$. The collection D separates points in X. In fact, let $x, y \in X$, different. Suppose that $f_m(x) = f_m(y)$, for each $m \in \mathbb{N}$. Since X is completely regular Hausdorff, there exists $g \in C_b(X)$ such that $g(x) = 1$ and $g(y) = 0$. Since $pw = \tau_{A(X)} \leqslant \gamma_{A(X)}$, we have that D is pw-dense in $C_b(X)$. Then, there exists $n \in \mathbb{N}$ such that $|1 - f_n(x)| = |g(x) - f_n| < \frac{1}{4}$ and $|f_n(x)| = |f_n(y)| = |g(y) - f_n(x)| < \frac{1}{4}$, which is impossible. Therefore, D separates points in X. We will prove that the topology in X is finer than some metrizable topology. By the previous, the function $d : X \times X \to \mathbb{R}$, defined by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n(1 + |f_n(x) - f_n(y)|)},$$

is a metric in X. Notice that the topology τ_d generated by d is the vector topology less fine in X which makes continuous every function f_n, with $n \in \mathbb{N}$. Then, τ_d is less fine than the topology in X. We will finalize the proof showing that (X,τ_d) is separable. In fact, the subsets of X of the form:

$$\bigcap_{k=1}^{m} f_{n_k}^{-1}(z_k - r_k, z_k + r_k)$$

where $m \in \mathbb{N}$, $\{z_1,\dots,z_m,r_1,\dots,r_m\} \subset \mathbb{Q}$ and $r_i > 0, i = 1,\dots,m$, form a countable base for the topology τ_d. Then, applying the Kakutani theorem, (a) is satisfied. $\square$

Proposition 2.5.12. *If a countable union of elements of $\mathcal{P}$ is dense in X and E is metrizable, then $(C_b(X,E),\beta_{\mathcal{P}})$ is an angelical space.*

Proof. By C.4.5 the space $(C(X,E),pw)$ of the continuous functions in X with values in E, equipped with the pointwise convergence topology, is angelical.

Considering the canonic inclusion $i : C_b(X,E) \to C(X,E)$. Since the topological vector spaces are regular, by C.4.2 we have that the space $(C_b(X,E),pw)$ is angelical. Since $pw \leqslant \tau_{\mathcal{P}} \leqslant \beta_{\mathcal{P}}$ the identity $id : C_b(X,E) \to C_b(X,E)$ is $(\beta_{\mathcal{P}},pw)$-continuous. Once again by C.4.2, we have that $(C_b(X,E),\beta_{\mathcal{P}})$ is an angelical space. $\square$

2.6. Completeness and compactness in $(C_b(X,E),\beta_{\mathcal{P}})$.

In this section we study the necessary and sufficient conditions to make the space $(C_b(X,E),\beta_{\mathcal{P}})$ be complete, quasicomplete and sequentially complete. Also, we will give a characterization of the relatively $\beta_{\mathcal{P}}$-compact sets.

To present the results of completeness, we need to develop some lemmas and previous definitions.

Lemma 2.6.1. *(Extension theorem for completely regular Hausdorff space.)*

Let X be a completely regular Hausdorff space and K a compact subspace of X. Each continuous function $f : K \to \mathbb{R}$, has a continuous extension $F : X \to \mathbb{R}$, such that $\sup\{|f(x)| : x \in K\} = \sup\{|F(x)| : x \in X\}$.

Proof. We can identify X with a topological subspace of the compactification of Stone-Čech βX, and therefore, we can identify K with a compact subspace of βX. Since βX is a normal space, by the extension theorem of Tietze, there exists a continuous function $G : \beta X \to \mathbb{R}$, such that $G|_K = f$ and $\sup\{|f(x)| : x \in K\} = \sup\{|G(x)| : x \in \beta X\}$. Then, it is enough to consider the function $F = G|_X$, which satisfies the requirements. □

Lemma 2.6.2. *Suppose that $(f_\alpha)_{\alpha\in\Lambda}$ is a net in $C_b(X,E)$ which is pw-convergent to a function $f : X \to E$.*

(a) If the net $(f_\alpha)_{\alpha\in\Lambda}$ is τ_u-Cauchy, then, it is τ_u-convergent to f and $f \in C_b(X,E)$.

(b) If the net $(f_\alpha)_{\alpha\in\Lambda}$ is $\beta_{\mathcal{P}}$-Cauchy, then, it is $\beta_{\mathcal{P}}$-convergent to f, $f \in B(X,E)$ and for all $K \in \mathcal{P}$, $f|_K : K \to E$ is continuous.

(c) If the net $(f_\alpha)_{\alpha\in\Lambda}$ is $\tau_{\mathcal{P}}$-Cauchy, then, given $p \in s(E)$, $K \in \mathcal{P}$ and $\varepsilon > 0$, there exists $\alpha_0 \in \Lambda$, such that: $\|f_\alpha - f\|_{p,K} < \varepsilon$, for each $\alpha \geqslant \alpha_0$. Also, for all $K \in \mathcal{P}$, $f|_K : K \to E$ is continuous.

Proof. (a) Given $p \in s(E)$ and $\varepsilon > 0$, there exists $\alpha_0 \in \Lambda$, such that: $\|f_\alpha - f_\beta\|_p < \varepsilon$, for each $\alpha,\beta \geqslant \alpha_0$. By B.1, we have that $\|f_\alpha - f\|_p < \varepsilon$, for each $\alpha \geqslant \alpha_0$, which implies that $\|f\|_p \leqslant \|f_\alpha\|_p + \varepsilon < \infty$. Applying B.2 we have that $p \circ f$ is continuous. Now by the arbitrariness of $\varepsilon > 0$ and $p \in s(E)$, it is concluded that the net $(f_\alpha)_{\alpha\in\Lambda}$ is τ_u-convergent to f and $f \in C_b(X,E)$.

(c) Given $p \in s(E)$, $K \in \mathcal{P}$ and $\varepsilon > 0$, there exists $\alpha_0 \in \Lambda$, such that: $\|f_\alpha - f_\beta\|_{p,K} < \varepsilon$, for each $\alpha,\beta \geqslant \alpha_0$. By the result B.1, we have that $\|f_\alpha - f\|_{p,K} < \varepsilon$, for each $\alpha \geqslant \alpha_0$.

Applying B.2, we have that $p \circ f|_K$ is continuous. Now by the arbitrariness of $p \in s(E)$ and of $K \in \mathcal{P}$, it is concluded that for each $K \in \mathcal{P}$, $f|_K$ is continuous.

(b) Given $x \in X$, we have:

$$\begin{aligned} v(x)p(f_\alpha(x) - f(x)) &\leqslant v(x)p(f_\alpha(x) - f_\beta(x)) + v(x)p(f_\beta(x) - f(x)) \\ &\leqslant \|v\|\|f_\alpha - f_\beta\|_{p,v} + \|v\|p(f_\beta(x) - f(x)) \ , \end{aligned}$$

Let $\varepsilon > 0$. By hypothesis, there exists $\alpha_0 \in \Lambda$, which satisfies:

$$v(x)p(f_\alpha(x) - f(x)) \leqslant \varepsilon + \|v\| p(f_\beta(x) - f(x)) \ ,$$

for all $x \in X$ and $\alpha, \beta \geqslant \alpha_0$. Consider a fix $y \in X$. For $r > 0$, there exist $\beta_0 \in \Lambda$, $\beta_0 \geqslant \alpha_0$, such that: $\|v\| p(f_{\beta_0}(y) - f(y)) \leqslant r$. Thereby, $v(y)p(f_\alpha(y) - f(y)) \leqslant \varepsilon + r$, where $r > 0$ is arbitrary, therefore $v(y)p(f_\alpha(y) - f(y)) \leqslant \varepsilon$. Since $y \in X$ is arbitrary, it is concluded that $\|f_\alpha - f\|_{p,v} \leqslant \varepsilon$, for each $\alpha \geqslant \alpha_0$. This implies that $\|f\|_{p,v} \leqslant \|f_\alpha\|_{p,v} + \varepsilon < \infty$. By the arbitrariness of $p \in s(E)$ and of $v \in V_{\mathcal{P}}$, alongside the result 2.2.2, it is concluded that $f \in B(X,E)$. Also, by the arbitrariness of $p \in s(E)$, $v \in V_{\mathcal{P}}$ and $\varepsilon > 0$, it is proved the $\beta_{\mathcal{P}}$-convergence of the net $(f_\alpha)_{\alpha\in\Lambda}$ to the function f. Therefore, the net is $\tau_{\mathcal{P}}$-convergent to f, and applying (c), it is concluded the result. □

Consider a topological space $\mathcal{X}$ and a covering P of $\mathcal{X}$.

Definition 2.6.3. *A topological space $\mathcal{X}$ is a **K_P-space** if it satisfies the condition: $f : \mathcal{X} \to \mathbb{R}$ is continuous, if and only if, for all $D \in P$, the restriction $f|_D$ is continuous.*

Lemma 2.6.4. *The following statements are equivalent:*

(a) $\mathcal{X}$ is a K_P-space.

(b) For each range bounded function $f : \mathcal{X} \to \mathbb{R}$, we have that f is continuous, if and only if, for all $D \in P$, the restriction $f|_D$ is continuous.

Proof. Is evidently that $(a) \Rightarrow (b)$. Now we will prove $(b) \Rightarrow (a)$. Suppose that (b) is verified. Let $f : \mathcal{X} \to \mathbb{R}$ be a function such that for each $D \in P$, the restriction $f|_D$ is continuous. Considering the homeomorphism $g : \mathbb{R} \to (-1,1)$, $g(x) = x/(1+|x|)$, the function $g \circ f$ is range bounded, such that for each $D \in P$, the restriction $g \circ f|_D$ is continuous. By hypothesis, $g \circ f$ is continuous and from this $f = g^{-1} \circ (g \circ f)$ is continuous. □

Proposition 2.6.5. *The space $(C_b(X,E),\beta_{\mathcal{P}})$ is complete (respectively quasicomplete) if and only if X is a $K_{\mathcal{P}}$-space and E is complete (respectively quasicomplete).*

Proof. Suppose that the space $(C_b(X, E), \beta_{\mathcal{P}})$ is complete (resp. quasicomplete). By the corollary 2.3.8, the spaces E and $(C_b(X), \gamma_{\mathcal{P}})$ are completes (resp. quasicompletes). Let $f : X \to \mathbb{R}$ be a range bounded function, such that for each $K \in \mathcal{P}$, the restriction $f|_K$ is continuous. By the extension theorem 2.6.1, for each $K \in \mathcal{P}$, there exists a extension $g_K \in C_b(X)$ of $f|_K$ such that $\sup\{|f|_K(x)| : x \in K\} = \|g_K\|$. Since $\mathcal{P}$ is a directed set with the relation $\subset$, we can consider the net $(g_K)_{K\in\mathcal{P}}$, which is $\tau_{\mathcal{P}}$-Cauchy.

Since the topologies $\gamma_{\mathcal{P}}$ and $\tau_{\mathcal{P}}$ coincide in the set $\{g \in C_b(X) : \|g\| \leqslant \|f\|\}$ (2.2.3), it is concluded that $(g_K)_{K\in\mathcal{P}}$ is a net $\gamma_{\mathcal{P}}$-Cauchy and therefore, is $\gamma_{\mathcal{P}}$-convergent to a function $h \in C_b(X)$. Since $(g_K)_{K\in\mathcal{P}}$ is a net τ_p-convergent to h, given $\widehat{K} \in \mathcal{P}$ and $\varepsilon > 0$, there exist $K_0 \in \mathcal{P}$, $K_0 \supset \widehat{K}$ such that, $\|g_K - h\|_{\widehat{K}} = \|f - h\|_{\widehat{K}} < \varepsilon$, for each $K \supset K_0$. By the arbitrariness of $\varepsilon > 0$, it follows that $f|_{\widehat{K}} = h|_{\widehat{K}}$ and by the arbitrariness of $\widehat{K}$ we have that $f = h \in C_b(X)$.

Finaly, applying the lemma 2.6.4, X is a $K_{\mathcal{P}}$-space.

Conversely, suppose that X is a $K_{\mathcal{P}}$-space and E is complete (resp. quasicomplete). Let $(f_\alpha)_{\alpha\in\Lambda}$ be a net $\beta_{\mathcal{P}}$-Cauchy in $C_b(X, E)$. Then is pw-Cauchy. Due to the completeness of E, we can define the function $f : X \to E$, $f(x) = \text{lím}\, f_\alpha(x)$. Applying the result 2.6.2.b we obtain that $f \in B(X, E)$, and for each $K \in \mathcal{P}$, the restriction $f|_K$ is continuous. Also, we have that the net $(f_\alpha)_{\alpha\in\Lambda}$ is $\beta_{\mathcal{P}}$-convergent to f. For $p \in s(E)$ the function $p \circ f|_K$ is continuaous for each $K \in \mathcal{P}$, thus the function $p \circ f$ is continuous, since X is a $K_{\mathcal{P}}$-space. By the arbitrariness of $p \in s(E)$, it is concluded that $f \in C_b(X, E)$. □

Definition 2.6.6. *A completely regular Hausdorff space X is a **P-space** if all elements of $\mathcal{Z}$ is open.*

Lemma 2.6.7. *The space X is a P-space, if and only if any G_δ subset of X is open.*

Proof. Suppose that X is a P-space. Let $U = \bigcap_{n=1}^{\infty} U_n$, where for each $n \in \mathbb{N}$, U_n is an open subset of X. Since every unitary subset of X is closed, given $x \in U$ and $n \in \mathbb{N}$, there exists $f_n \in C_b(X)$ such that $0 \leqslant f_n \leqslant \mathscr{X}_X$, $f_n(x) = 0$ and $f_n(X \backslash U_n) = \{1\}$. Then $x \in \bigcap_{n=1}^{\infty} f_n^{-1}(\{0\}) \subset U$, where $\bigcap_{n=1}^{\infty} f_n^{-1}(\{0\})$ is an open set, this by 1.3.2.2 and by hypothesis. Thereby it is concluded that U is a neighborhood of each of its elements.

Now we suppose any G_δ subset of X is open. Let $f \in C_b(X)$.

Since the set $f^{-1}(\{0\}) = \bigcap_{n=1}^{\infty}\{x \in X : |f(x)| < 1/n\}$, then this will be an open set. Therefore X is a P-space. $\square$

Proposition 2.6.8. *The space $(C_b(X), \gamma_{A(X)})$ is sequentially complete if and only if X is a P-space.*

Proof. Suppose that $(C_b(X), \gamma_{A(X)})$ is sequentially complete. Consider $Z \in \mathcal{Z}$ and $f \in C_b(X)$ such that $Z = f^{-1}(\{0\})$ and $0 \leqslant f \leqslant \mathscr{X}_X$. Note that the sequence $((1-f)^n)_{n\in\mathbb{N}}$ is pw-convergent to $\mathscr{X}_Z$ and by 2.2.3 is $\gamma_{A(X)}$-Cauchy. Then, by hypothesis we have $\mathscr{X}_Z \in C_b(X)$ and from this it is concluded that Z is open. Thus X is a P-space.

Now suppose that X is a P-space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence $\gamma_{A(X)}$-Cauchy in $C_b(X)$. We can define the function $f : X \to \mathbb{R}$, $f(x) = \text{lím}\, f_n(x)$. Applying the proposition 2.6.2.b, we have $f \in B(X)$ and the sequence $(f_n)_{n\in\mathbb{N}}$ is $\gamma_{A(X)}$-convergent to f. It is enough to prove that f is continuous. For this, we choose an arbitrary $x \in X$.

Given $\varepsilon > 0$, for each $n \in \mathbb{N}$, there exists an open neighborhood V_n of x, such that for each $y \in V_n$, $|f_n(y) - f_n(x)| < \varepsilon/3$. Since X is a P-space, by 2.6.7 we have that $V = \bigcap_{n=1}^{\infty} V_n$ is a neighborhood of x, such that for all $n \in \mathbb{N}$, and for each $y \in V$, $|f_n(y) - f_n(x)| < \varepsilon/3$. Thereby, for each $y \in V$, we have:

$$\begin{aligned} |f(x) - f(y)| &\leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< |f(x) - f_n(x)| + \varepsilon/3 + |f_n(y) - f(y)| \end{aligned}$$

for all $n \in \mathbb{N}$. Then, we choose $n_0 \in \mathbb{N}$, such that $|f(x) - f_{n_0}(x)| < \varepsilon/3$ and $|f_{n_0}(y) - f(y)| < \varepsilon/3$, proving that f is continuous in x. Thus, the sequence $(f_n)_{n\in\mathbb{N}}$ is $\gamma_{A(X)}$-convergent in $C_b(X)$. $\square$

In the following, we will denote by $\mathbf{\Gamma}$ the collection of all the families $\mathcal{P} \subset K(X)$ such that: $X = \bigcup\{K : K \in \mathcal{P}\}$ and for each pair of elements K_1 and K_2 of $\mathcal{P}$ there exists an element K_3 in $\mathcal{P}$, such that $K_1 \cup K_2 \subset K_3$.

Proposition 2.6.9. *If E is sequentially complete and X is a P-space, then, for each $\mathcal{P} \in \Gamma$, the space $(C_b(X,E), \beta_{\mathcal{P}})$ is sequentially complete.*

Proof. Let $\mathcal{P}\in\Gamma$ and consider $(f_n)_{n\in\mathbb{N}}$ a sequence $\beta_{\mathcal{P}}$-Cauchy in $C_b(X,E)$. Since E is sequentially complete, we can define the function $f: X\to E$, $f(x)=\text{lím}\, f_n(x)$. By proposition 2.6.2.b, is obtain that $f\in B(X,E)$, and the sequence $(f_n)_{n\in\mathbb{N}}$ is $\beta_{\mathcal{P}}$-convergent to f. It is enough to prove that f is continuous. For this, we choose and arbitrary $x\in X$. By the continuity of the functions f_n, by the lemma 2.6.7 and by the same argument used in 2.6.8, given $\varepsilon>0$ and $p\in s(E)$, there exists a neighborhood V of x, such that for all $n\in\mathbb{N}$, and for each $y\in V$, $p(f_n(y)-f_n(x))<\varepsilon/3$. For each $y\in V$, we have:

$$\begin{aligned}p(f(x)-f(y)) &\leqslant p(f(x)-f_n(x))+p(f_n(x)-f_n(y))+p(f_n(y)-f(y))\\ &< p(f(x)-f_n(x))+\varepsilon/3+p(f_n(y)-f(y))\end{aligned}$$

for every $n\in\mathbb{N}$. We choose $n_0\in\mathbb{N}$, such that

$$p(f(x)-f_{n_0}(x))<\varepsilon/3 \quad\text{and}\quad p(f_{n_0}(y)-f(y))<\varepsilon/3.$$

Thereby f is continuous in x, and from this, it follows that $f\in C_b(X,E)$. □

Proposition 2.6.10. *The space $(C_b(X,E),\delta)$ is sequentially complete, if and only if E is sequentially complete and X is a P-space.*

Proof. If $(C_b(X,E),\delta)$ is sequentially complete, then E and $(C_b(X),\gamma_{A(X)})$ are sequentially complete spaces, since subsets are δ-closed of $C_b(X,E)$ (2.3.7). Then, by 2.6.8, X is a P-space. The converse is immediately of 2.6.9. □

Corollary 2.6.11. *If $(C_b(X,E),\delta)$ is sequentially complete, then for each $\mathcal{P}\in\Gamma$, the space $(C_b(X,E),\beta_{\mathcal{P}})$ is sequentially complete.*

Proof. It is consequence of 2.6.10 and 2.6.9. □

The results 2.6.9, 2.6.10 and 2.6.11 were presented by J. Zafarani in [31]. We will strengthen this result showing an example, which there exists $\mathcal{P}\in\Gamma$ such that, the space $(C_b(X,E),\beta_{\mathcal{P}})$ is sequentially complete and, however, the space $(C_b(X,E),\delta)$ is not sequentially complete. To do this, we need to develop the following result.

Proposition 2.6.12.

(a) If X is a $K_{\mathcal{P}}$-space and E is sequentially complete, then for each $\mathcal{P}_1 \in \Gamma$, such that $\mathcal{P}_1 \preceq \mathcal{P}$, the space $(C_b(X,E),\beta_{\mathcal{P}_1})$ is sequentially complete.

(b) If $\mathcal{P}$ have countable cofinal subfamily and $(C_b(X,E),\beta_{\mathcal{P}})$ is sequentially complete, then the space X is a $K_{\mathcal{P}}$-space and E is sequentially complete.

Proof. (a) Note that for $\mathcal{P}_1 \in \Gamma$, such that $\mathcal{P}_1 \preceq \mathcal{P}$, X is also a $K_{\mathcal{P}_1}$-space. Let $(f_n)_{n\in\mathbb{N}}$ be sequence $\beta_{\mathcal{P}_1}$-Cauchy in $C_b(X,E)$. Since E is sequentially complete, we can define the function $f : X \to E$, $f(x) = \text{lím}\, f_n(x)$.

By proposition 2.6.2.b, it is obtained that $f \in B(X,E)$, and the sequence $(f_n)_{n\in\mathbb{N}}$ is $\beta_{\mathcal{P}_1}$-convergent to f. Also, for each $K \in \mathcal{P}_1$, the restriction $f|_K$ is continuous. Since X is a $K_{\mathcal{P}_1}$-space, it follows that $f \in C_b(X,E)$.

(b) Now suppose that $(C_b(X,E),\beta_{\mathcal{P}})$ is sequentially complete and that $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ is a cofinal subset of $\mathcal{P}$. Since E and $(C_b(X),\gamma_{\mathcal{P}})$ are closed subspaces of $(C_b(X,E),\beta_{\mathcal{P}})$, it follows that they are sequentially completes.

To prove that X is a $K_{\mathcal{P}}$-space we can consider a subfamily $\mathcal{Q} = \{Q_n : n \in \mathbb{N}\}$ of $\mathcal{K}$, defined as it follows: $Q_1 = K_1$, and for $n \geqslant 2$, it is defined recursively that $Q_n \in \mathcal{K}$, such that $Q_{n-1}, K_n \subset Q_n$, this is possible since $\mathcal{P}$ is directed and $\mathcal{K}$ is cofinal. Thereby $\mathcal{Q}$ is entirely arranged, directed and cofinal in $\mathcal{P}$. Let $f : X \to \mathbb{R}$ be a range bounded function, such that for each $K \in \mathcal{P}$, the restriction $f|_K$ is continuous. By the extension theorem 2.6.1, for each $Q_n \in \mathcal{Q}$, there exists an extension $g_n \in C_b(X)$ of $f|_{Q_n}$ such that $\sup\{|f|_{Q_n}(x)| : x \in Q_n\} = \|g_n\|$.

Consider the sequence $(g_n)_{n\in\mathbb{N}}$, which is $\tau_{\mathcal{P}}$-Cauchy. In fact, given $K \in \mathcal{P}$, there exists $n_0 \in \mathbb{N}$, such that $Q_m, Q_n \supset K$ for $m,n \geqslant n_0$. Then, $\|g_m - g_n\|_K = \|f - f\|_K = 0$ for $m,n \geqslant n_0$. Since the topologies $\gamma_{\mathcal{P}}$ and $\tau_{\mathcal{P}}$ coincide in the set $\{g \in C_b(X) : \|g\| \leqslant \|f\|\}$ (2.2.3), it is concluded that $(g_n)_{n\in\mathbb{N}}$ is a sequence $\gamma_{\mathcal{P}}$-Cauchy and therefore is $\gamma_{\mathcal{P}}$-convergent to a function $h \in C_b(X)$.

Given $\widehat{K} \in \mathcal{P}$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $Q_{n_0} \supset \widehat{K}$, and $\|g_n - h\|_{\widehat{K}} = \|f - h\|_{\widehat{K}} < \varepsilon$, for each $Q_n \supset Q_{n_0}$, this is, for each $n \geqslant n_0$. By the arbitrariness of $\varepsilon > 0$, it follows that $f|_{\widehat{K}} = h|_{\widehat{K}}$ and by the arbitrariness of $\widehat{K}$ we have that $f = h \in C_b(X)$.

Finally, applying the lemma 2.6.4, it is concluded that X is a $K_{\mathcal{P}}$-space. □

Remark 2.6.13. *In general, the fact that there exists $\mathcal{P} \in \Gamma$, such that $(C_b(X,E),\beta_{\mathcal{P}})$ is sequentially complete, it is not implied that $(C_b(X,E),\delta)$ is sequentially complete.*

Proof. Suppose that $X = \mathbb{R}$, $\mathcal{P} = \{[-n,n] : n \in \mathbb{N}\}$ and E is a locally convex and sequentially complete Hausdorff space. Then X is a $K_{\mathcal{P}}$-space.

Applying the result 2.6.12.a, it is obtain that $(C_b(X,E),\beta_{\mathcal{P}})$ is sequentially complete. However, X is not a P-space. Thereby, by 2.6.10, it is concluded that $(C_b(X,E),\delta)$ is not sequentially complete. □

In the following, we present a characterization of the Arzela-Ascoli Theorem $(C_b(X,E),\ \beta_{\mathcal{P}})$. Also, we will generalize one of the necessary and sufficient conditions presented in proposition 4.5 of [31].

Proposition 2.6.14. *Suppose that X is a $K_{\mathcal{P}}$-space and that E is quasicomplete. A subset H of $C_b(X,E)$ is relatively $\beta_{\mathcal{P}}$-compact, if and only if, it satisfies the following conditions:*

(a) H is τ_u-bounded.

(b) $H|_K := \{f|_K : f \in H\}$ is equicontinuous, for each $K \in \mathcal{P}$.

(c) For each $x \in X$, the set $H(x) = \{f(x) : f \in H\} \subset E$ is relatively compact.

Proof. Suppose that H is relatively $\beta_{\mathcal{P}}$-compact. By the proposition 2.2.2 it follows that H is τ_u-bounded. Since $\tau_{\mathcal{P}} \leqslant \beta_{\mathcal{P}}$, applying the theorem A.2, it is obtain that the $\tau_{\mathcal{P}}$-closure of H is $\tau_{\mathcal{P}}$-precompact, and from this, H is also $\tau_{\mathcal{P}}$-precompact. Thereby, by the Ascoli Theorem (A.1), we have that $H|_K$ is equicontinuous for each $K \in \mathcal{P}$, and that $H(x)$ is precompact for each $x \in X$. Since the closure of every subset in E precompact is precompact, applying the theorem A.2 we obtain (c).

Conversely, suppose that (a), (b) and (c) are satisfied. By the theorem A.1, we have that H is $\tau_{\mathcal{P}}$-precompact. Since $\tau_{\mathcal{P}}$ and $\beta_{\mathcal{P}}$ coincide in bounded sets, we have that H is also $\beta_{\mathcal{P}}$-precompact. In fact, let W be a $\beta_{\mathcal{P}}$-neighborhood of 0. Since $H - H$ is a τ_u-bounded that contains 0, by 2.2.3, there exists a $\tau_{\mathcal{P}}$-neighborhood of 0, such that $V \cap (H - H) \subset W \cap (H - H)$. Since H is $\tau_{\mathcal{P}}$-precompact, there exist $f_1, \ldots, f_n$ in H, such that $H \subset \bigcup_{i=1}^{n}(f_i + V)$. For each $i \in \{1, \ldots, n\}$, we define the set $H_i = \{f \in H : f - f_i \in V\}$. Then we have $H = \bigcup_{i=1}^{n} H_i$ y $\bigcup_{i=1}^{n}(H_i - f_i) \subset V$.

Thereby, for each $i = 1, \ldots, n$, we have:

$$H_i - f_i \subset \bigcup_{i=1}^{n}(H_i - f_i) \cap (H - H) \subset V \cap (H - H) \subset W \cap (H - H) \subset W \ .$$

Finally,

$$H = \bigcup_{i=1}^{n} H_i \subset \bigcup_{i=1}^{n}(f_i + W)$$

Furthermore, the space $(C_b(X,E), \beta_{\mathcal{P}})$ is quasicomplete, this is by 2.6.5. From this, we have that H is relatively $\beta_{\mathcal{P}}$-complete.

Finally, by the theorem A.2, it is concluded that H is relatively $\beta_{\mathcal{P}}$-compact. $\square$

2.7. Other characterizations of β_p in $C_b(X,E)$.

In this section, we will fix an arbitrary seminorm $p \in s(E)$ and characterize the topology β_p on $C_b(X,E)$, through various bases of neighborhoods of 0, which will facilitate the work in the following chapters to characterize the $\beta_{\mathcal{P}}$-continuity of the operator and functionals in $C_b(X,E)$ and its characterization as integrals.

One of the most interesting results of this section, is that the topology β_p is the finest vector topology which coincide with the topology τ_p in the u_p-bounded sets, or equivalently, that the topology β_p coincide with the mixed topology. $\gamma\,[u_p, \tau_p]$.

All the concepts and results concerning mixed topologies used in this section are available in Appendix D, based on the paper [30] by A. Wiweger.

It is shown in [30, example D, page 65] that if X is a Hausdorff completely regular space, then in $C_b(X)$ the vector topology $\gamma\left[\tau_{\|\cdot\|},\tau_p\right]$ has a base of neighborhoods of 0 constituted by all the sets of the form:

$$\bigcap_{i=1}^{\infty}\{f\in C_b(X):\|f\|_{K_i}\leqslant a_i\}$$

where $K_i\in\mathcal{P}$ and $0<a_i\to+\infty$.

Analogously, we will prove the following generalization:

Lemma 2.7.1. *Given $p\in s(E)$, in $C_b(X,E)$ the mixed topology $\gamma\left[u_p,\tau_p\right]$ has a base of neighborhoods 0 consisting of all the sets of the form:*

$$W(K_i,a_i)=\bigcap_{i=1}^{\infty}\{f\in C_b(X,E):\|f\|_{p,K_i}\leqslant a_i\}$$

where $K_i\in\mathcal{P}$ and $0<a_i\to+\infty$.

Proof. First we well prove that the topologies u_p and τ_p satisfy the conditions of the theorem D.10. For $f\in C_b(X,E)$, we have $\|f\|_p=\sup\limits_{K\in\mathcal{P}}\|f\|_{p,K}$. Let $K_1,\ldots,K_n\in\mathcal{P}$, $f\in C_b(X,E)$ and $\varepsilon>0$. Let $K\in\mathcal{P}$, such that $\bigcup_{i=1}^{n}K_i\subset K$ and $d:=\|f\|_{p,K}+\varepsilon$. The set $V=\{x\in X:p(f(x))<d\}$ is open and $K\subset V$. Sice F is compact, by 2.4.1, there exists $g\in C_b(X)$, such that $0\leqslant g\leqslant\mathscr{X}_X$, $g(K)=\{0\}$ and $g(X\backslash V)=\{1\}$. Defining $f_1=gf$ and $f_2=(1-g)f$. Then $f=f_1+f_2$, $\|f_2\|_{p,K_i}=0$ for $i=1,\ldots,n$ and $\|f_1\|_p\leqslant d$. Then, the result is obtained applying the theorem D.10. □

Also, by D.7, we have the following lemma.

Lemma 2.7.2. *The topology $\gamma\left[u_p,\tau_p\right]$ is the finest vector topology in $C_b(X,E)$ which coincides with the topology τ_p in the sets u_p-bounded.*

Proposition 2.7.3. *The topologies β_p and $\gamma\left[u_p,\tau_p\right]$ coincide in $C_b(X,E)$.*

Proof. By the previous lemma alongside with 2.2.4 it is obtained that $\beta_p \leqslant \gamma\,[u_p,\tau_p]$. Let $U = W(K_i,a_i)$ be a basic $\gamma\,[u_p,\tau_p]$-neighborhood of 0. We define the function $w : X \to \mathbb{R}$, by

$$w(x) = \begin{cases} \sup\limits_{f\in U}\{p(f(x))\} & , \text{ si } x \in \bigcup_{i=1}^{\infty} K_i \\ 0 & , \text{ si } x \notin \bigcup_{i=1}^{\infty} K_i \ . \end{cases}$$

This function is well defined, in fact, given $x \in \bigcup_{i=1}^{\infty} K_i$ there exists i, such that $x \in K_i$ and $U \subset \{f \in C_b(X,E) : \|f\|_{p,K_i} \leqslant a_i\}$. From this, it is concluded that $w(x) \leqslant a_i$.

We choose $N \in \mathbb{N}$, such that $a_n \geqslant 1$ for $n \geqslant N$ and we define $a := \text{mín}\{1, a_1, \ldots, a_N\}$. Defining the function $v : X \to [0,\infty)$ by

$$v(x) = \begin{cases} \dfrac{2}{w(x)+a} & , \text{ si } x \in \bigcup_{i=1}^{\infty} K_i \\ 0 & , \text{ si } x \notin \bigcup_{i=1}^{\infty} K_i \ . \end{cases}$$

We will prove that $v \in V_{\mathcal{P}}$. The function v is range bounded. In fact, for each $x \in X$, $v(x) \leqslant \frac{2}{a}$. In the following, we will prove that v $\mathcal{P}$-vanishes at infinite. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$, such that $\frac{1}{a_n} < \frac{\varepsilon}{4}$, for each $n \geqslant M$. Then,

$$\frac{2}{w(x)+a} \geqslant \varepsilon \Rightarrow \frac{2}{\varepsilon} \geqslant w(x) + a \geqslant w(x) \ .$$

Therefore, $\{x \in X : v(x) \geqslant \varepsilon\} \subset \{x \in X : w(x) \leqslant \frac{2}{\varepsilon}\}$. Also

$$\{x \in X : w(x) \leqslant \frac{2}{\varepsilon}\} \subset \bigcup_{i=1}^{M} K_i \ .$$

In fact, suppose that there exists $x \in X$, such that $w(x) \leqslant \frac{2}{\varepsilon}$ and $x \notin \bigcup_{i=1}^{M} K_i$. There exists $f \in C_b(X)$, such that $0 \leqslant f \leqslant \mathscr{X}_X$, $f(x) = 1$ and $f(\bigcup_{i=1}^{M} K_i) = \{0\}$. We can choose $e \in E$, such that $p(e) = \frac{4}{\varepsilon}$. Then, $f \otimes e$ is such that $\|f \otimes e\|_p \leqslant \frac{4}{\varepsilon}$ and $p(f(x)e) = \frac{4}{\varepsilon}$. Since $f \otimes e = 0$ in $\bigcup_{i=1}^{M} K_i$ and $\frac{4}{\varepsilon} < a_n$, for $n \geqslant M$, we have that $f \otimes e \in U$. From the previous, $\frac{4}{\varepsilon} = p(f \otimes e(x)) \leqslant w(x) \leqslant \frac{2}{\varepsilon}$, which is a contradiction.

Then, since there exists $K \in \mathcal{P}$, such that $\bigcup_{i=1}^{M} K_i \subset K$, it is concluded that $\{x \in X : v(x) \geqslant \varepsilon\} \subset K$, proving that $\|v\|_{X\setminus K} \leqslant \varepsilon$ and therefore $v \in V_{\mathcal{P}}$.

Finally, it is affirmed that $\{f \in C_b(X,E) : \|f\|_{p,v} \leqslant 1\} \subset U$. In fact, first note that for $x \in K_i$, we have: $a_i \geqslant \frac{a_i+a}{2} \geqslant \frac{w(x)+a}{2}$ and from this $\frac{1}{a_i} \leqslant \frac{2}{w(x)+a}$. Then, if $\|f\|_{p,v} \leqslant 1$, given $i \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{1}{a_i}\|f\|_{p,K_i} = \sup_{x\in K_i} \frac{1}{a_i} p(f(x)) \leqslant \sup_{x\in K_i} \frac{2}{w(x)+a} p(f(x)) &\leqslant \sup_{x\in\bigcup_{i=1}^{\infty} K_i} \frac{2}{w(x)+a} p(f(x)) \\
&\leqslant \sup_{x\in X} v(x)p(f(x)) \\
&= \|f\|_{p,v} \\
&\leqslant 1
\end{aligned}$$

Thereby, $f \in U$. Therefore $\gamma\left[u_p, \tau_p\right] \leqslant \beta_p$. □

Corollary 2.7.4. *The topology β_p is the finest topological vector in $C_b(X,E)$ which coincides with τ_p in all the subsets u_p-bounded. Also the collection of all sets of the form:*

$$W(K_i, a_i) = \bigcap_{i=1}^{\infty} \{f \in C_b(X,E) : \|f\|_{p,K_i} \leqslant a_i\}$$

where $K_i \in \mathcal{P}$ and $0 < a_i \to +\infty$, is a base of neighborhoods of 0 for the topology β_p.

Proof. Is immediate from the results 2.7.1, 2.7.2 and 2.7.3. □

Proposition 2.7.5. *In $C_b(X,E)$ the subsets family*

$$\mathcal{V} = \left\{ U \subset C_b(X,E) : \begin{array}{c} U \text{ is balanced and absorbing such that} \\ \text{for each } \varepsilon > 0, \text{ there exists } V \text{ to } \tau_p\text{-neighborhood of } 0 \\ \text{which satisfies } V \cap \{f \in C_b(X,E) : \|f\|_p \leqslant \varepsilon\} \subset U \end{array} \right\}$$

is a base of neighborhoods of 0 for the topology β_p.

Proof. The family $\mathcal{V}$ determines a unique locally convex topology τ in $C_b(X,E)$ that has $\mathcal{V}$ as the base of neighborhoods of 0. In fact, is clear that the finite intersection of elements of $\mathcal{V}$ belongs to $\mathcal{V}$. Let $\alpha \in \mathbb{K}, \alpha \neq 0$ and $U \in \mathcal{V}$. Given $\varepsilon > 0$, there exists V, a τ_p-neighborhood of 0, such that $V \cap \{f \in C_b(X,E) : \|f\|_p \leqslant \frac{\varepsilon}{|\alpha|}\} \subset U$. Then, $\alpha V \cap \{f \in C_b(X,E) : \|f\|_p \leqslant \varepsilon\} \subset \alpha U$, proving that $\alpha U \in \mathcal{V}$.

Each balanced τ_p-neighborhood of 0 belongs to $\mathcal{V}$, therefore $\tau_p \leqslant \tau$. Also, by the definition of $\mathcal{V}$, we have $\tau \leqslant \tau_p$, in the sets of the form $\{f \in C_b(X,E) : \|f\|_p \leqslant \varepsilon\}$. From this, it is concluded that the topologies τ and τ_p coincide in the u_p-bounded subsets of $C_b(X,E)$.

Let $\tau*$ be a vector topology in $C_b(X,E)$ which coincide with τ_p in the u_p-bounded sets. If U is a balanced $\tau*$-neighborhood of 0, then, for each $\varepsilon > 0$, there exists V, a τ_p-neighborhood of 0, such that $V \cap \{f \in C_b(X,E) : \|f\|_p \leqslant \varepsilon\} \subset U \cap \{f \in C_b(X,E) : \|f\|_p \leqslant \varepsilon\}$. Therefore $U \in \mathcal{V}$ and thus $\tau* \leqslant \tau$. Thereby, τ is the finest topological vector in $C_b(X,E)$ which coincides with τ_p in the u_p-bounded sets. Applying 2.7.4, we have $\tau = \beta_p$. □

Chapter 3

Operators and functionals in $C_b(X, E)$.

3.1. The $\beta_{\mathcal{P}}$-continuity of operators and functionals in $C_b(X, E)$.

In this section, we will characterize the $\beta_{\mathcal{P}}$-continuity of operators and linear functionals on $C_b(X, E)$.

Let H and F be two Hausdorff locally convex spaces and consider q a continuous seminorm in F. To say that an operator $T : H \to F$ is $(\tau_H, \sigma(F, \{q\}))$-continuous, we will simply say that is (τ_H, q)-continuous. Also, to say that a net in F is $\sigma(F, \{q\})$-convergent, we will say that is q-convergent.

Definition 3.1.1. *Let F be a Hausdorff locally convex space. Given $p \in s(E)$ and $q \in s(F)$, a linear operator $T : C_b(X, E) \to F$ is* ***$\mathcal{P}_p^q$-tight*** *if the restriction $T|_B$ is (τ_p, q)-continuous, where $B = \{f \in C_b(X, E) : \|f\|_p \leqslant 1\}$ and τ_p is the subspace topology in B induced by the topology τ_p defined in $C_b(X, E)$.*

In case that $F = \mathbb{K}$ and $s(F) = \{q\} = \{|\cdot|\}$, to say a linear functional $T : C_b(X, E) \to \mathbb{K}$ is $\mathcal{P}_p^q$-tight, we will simply say that T is ***$\mathcal{P}_p$-tight****.*

Also, if E is a normed space where $s(E) = \{p\}$, then to refer to a linear functional $\mathcal{P}_p$-tight, we will simply say that is ***$\mathcal{P}$-tight****.*

By the different definitions of continuity for a linear operator, we have the following result.

Lemma 3.1.2. *Let $T : C_b(X, E) \to F$ be a linear operator, $p \in s(E)$ and $q \in s(F)$. The following statements are equivalent:*

(a) T is $\mathcal{P}_p^q$-tight.

(b) Given $\varepsilon > 0$, there exist $\delta > 0$ and $K \in \mathcal{P}$ such that: $q(T(f)) < \varepsilon$, for each $f \in C_b(X, E)$, such that $\|f\|_p \leqslant 1$ and $\|f\|_{p,K} \leqslant \delta$.

(c) For each net $(f_\alpha)_{\alpha\in\Lambda}$ in $C_b(X, E)$ such that, $\|f_\alpha\|_p \leqslant 1$, for all $\alpha \in \Lambda$ and $\|f_\alpha\|_{p,K} \to 0$ for each $K \in \mathcal{P}$, we have $(T(f_\alpha))_{\alpha\in\Lambda}$ is q-convergent to 0.

In the following we will give a generalization of the Theorem 3.2 presented in [8, page 846] by Robert Fontenot, which only refers to the functional on $(C_b(X, E), \beta_0)$ when E is normed.

Theorem 3.1.3. *Let $T : C_b(X, E) \to F$ be a linear operator. Consider the following propositions:*

1. *T is $(\beta_{\mathcal{P}}, \tau_F)$-continuous.*

2. *For all $q \in s(F)$, there exists $p \in s(E)$ such that T is (β_p, q)-continuous.*

3. *For all $q \in s(F)$, there exists $p \in s(E)$ such that T is $\mathcal{P}_p^q$-tight.*

4. *Given $\varepsilon > 0$ and $q \in s(F)$ there exist $K \in \mathcal{P}$ and $p \in s(E)$ such that $q(T(f)) < \varepsilon$ for each $f \in C_b(X, E)$ which satisfies $\|f\|_p \leqslant 1$ and $f|_K = 0$.*

5. *Any net $(f_\alpha)_{\alpha\in\Lambda}$ in $\{f \in C_b(X) : \|f\| \leqslant 1\}$, such that is $\tau_{\mathcal{P}}$-convergent to 0, satisfy the following condition: for each $q \in s(F)$, there exists $p \in s(E)$, such that $(T(f_\alpha g))_{\alpha\in\Lambda}$ is q-convergent to 0 uniformly for $g \in \{f \in C_b(X, E) : \|f\|_p \leqslant 1\}$.*

6. *Assuming that $\mathbb{K} = F = \mathbb{R}$, the map $M : C_b(X) \to \mathbb{R}$ defined by $M(f) = \sup\{|T(g)| : g \in C_b(X, E)$ such that $\forall p \in s(E), \forall x \in X, p(g(x)) \leqslant f(x)\}$ to $f \geqslant 0$ and defined by $M(f) := M(f^+) - M(f^-)$ for $f \in C_b(X)$, is a linear functional $\mathcal{P}_p$-tight.*

Are satisfied the following statements:

(a) The propositions 1, 2 *and* 3 *are equivalents.*

(b) The proposition 3 *implies the proposition* 4.

(c) By assuming that $\mathbb{K} = \mathbb{R}$ *and* T *is* (τ_u, τ_F)*-continuous, we have:* $4 \Rightarrow 5 \Rightarrow 6$.

(d) By assuming that $\mathbb{K} = \mathbb{R}$ *and* E *is normed with norm* p*, we have:* $6 \Rightarrow 1$.

Proof. $1 \Rightarrow 2$: If T is $(\beta_{\mathcal{P}}, \tau_F)$-continuous, then, given $\varepsilon > 0$ and $q \in s(F)$, there exist $\delta > 0$, $p_1, \ldots, p_n \in s(E)$ and $v_1, \ldots, v_n \in V_{\mathcal{P}}$, such that $q(T(f)) < \varepsilon$, for each $f \in \{g \in C_b(X, E) : \|g\|_{p_i, v_i} < \delta, \forall i = 1, \ldots, n\}$. Since $s(E)$ and $V_{\mathcal{P}}$ are directed families there exist $p \in s(E)$ and $v \in V_{\mathcal{P}}$ such that $p_i \leqslant p$ and $v_i \leqslant v$ for each $i \in \{1, \ldots, n\}$. Thus, $\|f\|_{p,v} < \delta \Rightarrow q(T(f)) < \varepsilon$, proving the (β_p, q)-continuity of T.

$2 \Rightarrow 3$: Let $(f_\alpha)_{\alpha \in \Lambda}$ be a net in $\{f \in C_b(X, E) : \|f\|_p \leqslant 1\}$ that is τ_p-convergent to 0. Then, by 2.2.4, the net is β_p-convergent to 0. Finally by the continuity of T we have $(T(f_\alpha))$ is q-convergent to 0. Therefore T is $\mathcal{P}_p^q$-tight.

$3 \Rightarrow 1$: Let $q \in s(F)$. There exists $p \in s(E)$ such that T is $\mathcal{P}_p^q$-tight. Consider $W = \{f \in C_b(X, E) : q(T(f)) \leqslant 1\}$. Applying 3.1.2, given $r > 0$, there exist $K \in \mathcal{P}$ and $\delta > 0$ such that: $q(T(f)) \leqslant \frac{1}{r}$ for each $f \in C_b(X, E)$ which satisfies $\|f\|_p \leqslant 1$ and $\|f\|_{p,K} \leqslant \delta$. Thereby

$$\{f \in C_b(X, E) : \|f\|_{p,K} \leqslant \delta r\} \cap \{f \in C_b(X, E) : \|f\|_p \leqslant r\} \subset W$$

proving that W is a β_p-neighborhood of 0; this by 2.7.5. Thus, T is (β_p, q)-continuous and therefore $(\beta_{\mathcal{P}}, q)$-continuous. The conclusion is obtained by the arbitrariness of q in $s(F)$.

$3 \Rightarrow 4$: Given $q \in s(F)$ and $\varepsilon > 0$, there exist $p \in s(E)$ and $K \in \mathcal{P}$ such that $q(T(f)) < \varepsilon$, for each $f \in C_b(X, E)$ which satisfies $\|f\|_p \leqslant 1$ and $\|f\|_{p,K} \leqslant \delta$. In particular, for $f \in C_b(X, E)$ such that $\|f\|_p \leqslant 1$ and $f|_K = 0$. Thereby, we have proved (b).

We will prove 4 $\Rightarrow$ 5 $\Rightarrow$ 6, assuming that $\mathbb{K} = \mathbb{R}$ and T is $(\tau_u, \tau_{_F})$-continuous. For 4 $\Rightarrow$ 5, by (a), given $q \in s(F)$, there exists $p \in s(E)$ such that the operator T is (u_p, q)-continuous. Thus, $\|T\|_p^q := \sup\{q(T(f)) : \|f\|_p \leqslant 1\} < \infty$. Suppose that 4 is satisfied and consider a net $(f_\alpha)_{\alpha\in\Lambda}$ in $\{f \in C_b(X) : \|f\| \leqslant 1\}$, that $\tau_{\mathcal{P}}$-converges to 0. We can assume that $f_\alpha \geqslant 0$ for each $\alpha \in \Lambda$. Given $\varepsilon > 0$, we choose $K \in \mathcal{P}$ mentioned in 4. Let $\alpha_0 \in \Lambda$ be such that $\|f_\alpha\|_K < \varepsilon' := \text{mín}\{1, \varepsilon\}$, for each $\alpha \geqslant \alpha_0$. Defining $h_\alpha = \text{mín}\{f_\alpha, \varepsilon'\}$. If $g \in \{f \in C_b(X, E) : \|f\|_p \leqslant 1\}$, then by 4, we have $q(T(f_\alpha g - h_\alpha g)) < \varepsilon$ for each $\alpha \geqslant \alpha_0$, since $f_\alpha - h_\alpha = 0$ in K. Then, for $\alpha \geqslant \alpha_0$, we have:

$$q(T(f_\alpha g)) < \varepsilon + q(T(h_\alpha g)) \leqslant \varepsilon + \|T\|_p^q \|h_\alpha g\|_p < \varepsilon + \|T\|_p^q \varepsilon$$

whichever is $g \in \{f \in C_b(X, E) : \|f\|_p \leqslant 1\}$.

Now proving 5 $\Rightarrow$ 6. Suppose $\mathbb{K} = F = \mathbb{R}$. As previously assumed, T is, in this case, a functional τ_u-continuous in $C_b(X, E)$, there exists $p_1 \in s(E)$ such that T is u_{p_1}-continuous. Suppose that 5 is satisfied, that is to say, for each net $(f_\alpha)_{\alpha\in\Lambda}$ in $\{f \in C_b(X) : \|f\| \leqslant 1\}$ that $\tau_{\mathcal{P}}$-converges to 0, there exists $p_2 \in s(E)$ such that $(T(f_\alpha g))_{\alpha\in\Lambda}$ is convergent to 0 uniformly for $g \in \{f \in C_b(X, E) : \|f\|_{p_2} \leqslant 1\}$. We choose $p' \in s(E)$, to comply $p_1, p_2 \leqslant p'$.

Then, for each net $(f_\alpha)_{\alpha\in\Lambda}$ in $\{f \in C_b(X) : \|f\| \leqslant 1\}$ that $\tau_{\mathcal{P}}$-converges to 0, there exists $p' \in s(E)$ such that T is $u_{p'}$-continuous and $(T(f_\alpha g))_{\alpha\in\Lambda}$ is convergent to 0 uniformly for $g \in \{f \in C_b(X, E) : \|f\|_{p'} \leqslant 1\}$.

We will prove that the map M mentioned in 6 is linear. Let $f, g \in C_b(X)$, $f, g \geqslant 0$. Clearly $M(af) = aM(f)$, for each $a \in \mathbb{R}$. Consider $h \in C_b(X, E)$ such that $\forall p \in s(E), p \circ h \leqslant f + g$ and defining the functions:

$$h_1 : X \to E, \quad h_1(x) = \begin{cases} \dfrac{f(x)}{f(x) + g(x)} \cdot h(x) & , \text{ if } f(x) + g(x) > 0 \\ 0 & , \text{ if } f(x) + g(x) = 0 \end{cases}$$

and

$$h_2 : X \to E, \quad h_2(x) = \begin{cases} \dfrac{g(x)}{f(x)+g(x)} \cdot h(x) & , \text{ if } f(x)+g(x) > 0 \\ 0 & , \text{ if } f(x)+g(x) = 0 \ . \end{cases}$$

Note that $p \circ h_1 \leqslant f$ and $p \circ h_2 \leqslant g$, for each $p \in s(E)$. Therefore $h_1, h_2 \in B(X, E)$. If $y \in X$ is such that $f(y)+g(y) > 0$, then, it is clear that h_1 and h_2 are continuous in y. If $y \in X$ is such that $f(y)+g(y) = 0$, then, $f(y) = g(y) = 0$. By the continuity of f and g, given $\varepsilon > 0$, the sets $A_\varepsilon = f^{-1}(-\varepsilon, \varepsilon)$ and $B_\varepsilon = g^{-1}(-\varepsilon, \varepsilon)$ are neighborhoods of y. Then, for each $p \in s(E)$ and $x \in A_\varepsilon$, we have $p(h_1(x) - h_1(y)) = p(h_1(x)) \leqslant |f(x)| < \varepsilon$. Furthermore, for each $p \in s(E)$ and $x \in B_\varepsilon$, we have $p(h_2(x) - h_2(y)) = p(h_2(x)) < \varepsilon$. Thereby $h_1, h_2 \in C_b(X, E)$. Since $|T(h)| = |T(h_1) + T(h_2)| \leqslant M(f) + M(g)$, it follows that $M(f+g) \leqslant M(f) + M(g)$.

For the other inequality, take $\varepsilon > 0$ and $h_1, h_2 \in C_b(X, E)$, such that $p \circ h_1 \leqslant f$, $p \circ h_2 \leqslant g$ for each $p \in s(E)$ and $0 \leqslant T(h_1) \leqslant M(f) \leqslant T(h_1) + \frac{\varepsilon}{2}$ alongside with $0 \leqslant T(h_2) \leqslant M(g) \leqslant T(h_2) + \frac{\varepsilon}{2}$. Then, $M(f) + M(g) \leqslant T(h_1) + T(h_2) + \varepsilon = T(h_1 + h_2) + \varepsilon \leqslant M(f+g) + \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we have $M(f)+M(g) \leqslant M(f+g)$ and from this it is concluded that $M(f) + M(g) = M(f+g)$.

Given $f \in C_b(X)$, consider $g, h \in C_b(X)$, $g, h \geqslant 0$ such that $f = g - h$. Then,

$$\begin{aligned} f^+ - f^- = g - h &\Rightarrow M(f^+) + M(h) = M(f^-) + M(g) \\ &\Rightarrow M(f) = M(g) - M(h) \end{aligned}$$

proving that M is well defined and linear in $C_b(X)$.

It now remains to prove that M is $\mathcal{P}$-tight. Let $(f_\alpha)_{\alpha \in \Lambda}$ be a net in $\{f \in C_b(X) : \|f\| \leqslant 1\}$ that $\tau_{\mathcal{P}}$-converges to 0. We can assumed that $f_\alpha \geqslant 0$, for each $\alpha \in \Lambda$. By 5, there exists $p' \in s(E)$ such that T is $u_{p'}$-continuous and $(T(f_\alpha g))_{\alpha \in \Lambda}$ is convergent to 0 uniformly for $g \in \{f \in C_b(X, E) : \|f\|_{p'} \leqslant 1\}$. Let $\varepsilon > 0$ and $\alpha_0 \in \Lambda$ such that $|T(f_\alpha g)| < \varepsilon$, for all $g \in \{f \in C_b(X, E) : \|f\|_{p'} \leqslant 1\}$ and for each $\alpha \geqslant \alpha_0$.

Let a fix $\alpha \geqslant \alpha_0$. Considering $g \in C_b(X, E)$ such that $p \circ g \leqslant f_\alpha$, for all $p \in s(E)$.

Then, we define the function

$$h : X \to E, \quad h(x) = \begin{cases} \dfrac{g(x)}{f_\alpha(x)} & , \text{ if } f_\alpha(x) > 0 \\ 0 & , \text{ if } f_\alpha(x) = 0 \ . \end{cases}$$

This function is bounded, in fact, $p(h(x)) \leqslant \dfrac{p(g(x))}{f_\alpha(x)} \leqslant 1$, for each $p \in s(E)$ and for all $x \in X$. In particular $\|h\|_{p'} \leqslant 1$. If $y \in X$, is such that, $f_\alpha(y) > 0$, then is clear that h is continuous in y. If $y \in X$, is such that, $f_\alpha(y) = 0$, then $h(y) = 0$. Given $\varepsilon > 0$, the set $C_\varepsilon = f_\alpha^{-1}(-\varepsilon, \varepsilon)$ is a neighborhood of y such that $p(h(x) - h(y)) = p(h(x)) \leqslant f_\alpha(x) < \varepsilon$, for all $x \in C_\varepsilon$ and each $p \in s(E)$. Therefore, $h \in C_b(X,E)$. Then, $f_\alpha h \in \{f \in C_b(X,E) : |T(f)| \leqslant \varepsilon\}$ and since T is $u_{p'}$-continuous, there exists $\delta > 0$ such that $f_\alpha h + \{f \in C_b(X,E) : \|f\|_{p'} < \delta\} \subset \{f \in C_b(X,E) : |T(f)| \leqslant \varepsilon\}$. If $f_\alpha(x) > 0$, then $f_\alpha(x)h(x) - g(x) = 0$, therefore $\|f_\alpha h - g\|_{p'} = \sup\{p'(f_\alpha(x)h(x) - g(x)) : f_\alpha(x) = 0\} \leqslant \sup\{p'(g(x)) : f_\alpha(x) = 0\} = 0$, the latter due to $p' \circ g \leqslant f_\alpha$. Therefore, $\|f_\alpha h - g\|_{p'} < \delta$ and from this $|T(g)| < \varepsilon$.

By the arbitrariness of g, it follows that $M(f_\alpha) \leqslant \varepsilon$, for each $\alpha \geqslant \alpha_0$. That is to say, M is $\mathcal{P}$-tight.

Now we will prove that (d) is satisfied. Suppose that $\mathbb{K} = F = \mathbb{R}$ and E is normed with norm p. If 6 is satisfied, then M is a linear functional $\mathcal{P}$-tight.

Applying (a), it follows that M is $\gamma_{\mathcal{P}}$-continuous in $C_b(X)$. Thereby there exists $v \in V_{\mathcal{P}}$ such that for each $\varepsilon > 0$, we have $\{f \in C_b(X,E) : \|f\|_v < \varepsilon\} \subset \{f \in C_b(X,E) : |M(f)| < \varepsilon\}$. Therefore, for each $f \in C_b(X,E)$, it is satisfied that $|M(f)| \leqslant \|f\|_v$. Let $f \in C_b(X,E)$. We have: $|T(f)| \leqslant M(p \circ f) \leqslant \|p \circ f\|_v = \|f\|_{p,v}$, therefore the linear functional T is $\beta_{\mathcal{P}}$-continuous, satisfying 1. □

As a direct consequence from 3.1.3.a, we have the following corollary:

Corollary 3.1.4. $\mathscr{L}^{\tau_F}_{\beta_{\mathcal{P}}}(C_b(X,E),F) = \bigcap_{q \in s(F)} \mathscr{L}^{q}_{\beta_{\mathcal{P}}}(C_b(X,E),F)$

$$= \bigcap_{q \in s(F)} \left(\bigcup_{p \in s(E)} \mathscr{L}^{q}_{\beta_p}(C_b(X,E),F) \right).$$

In fact, by the proof 3.1.3.a, we can state the following result.

Corollary 3.1.5. *Let $T : C_b(X, E) \to F$ be a linear map, $p \in s(E)$ and $q \in s(F)$.*

(a) The map T, is (β_p, q)-continuous, if and only if, is $\mathcal{P}_p^q$-tight.

In case that $F = \mathbb{K}$, we have the following equivalence.

(b) The functional T, is β_p-continuous, if and only if, is $\mathcal{P}_p$-tight.

Analogously, for linear functionals on $C_b(X, E)$, we have the following result:

Corollary 3.1.6. *Let $T : C_b(X, E) \to \mathbb{K}$ be a linear functional. Consider the following propositions:*

1. *T is $\beta_{\mathcal{P}}$-continuous.*
2. *There exists $p \in s(E)$ such that T is β_p-continuous.*
3. *There exists $p \in s(E)$ such that T is $\mathcal{P}_p$-tight.*
4. *Given $\varepsilon > 0$, there exist $K \in \mathcal{P}$ and $p \in s(E)$ such that $|T(f)| < \varepsilon$ for each $f \in C_b(X, E)$ which satisfies $\|f\|_p \leqslant 1$ and $f|_K = 0$.*
5. *Any net $(f_\alpha)_{\alpha\in\Lambda}$ in $\{f \in C_b(X) : \|f\| \leqslant 1\}$, such that is $\tau_{\mathcal{P}}$-convergent to 0, satisfy the following condition: there exists $p \in s(E)$ such that $(T(f_\alpha g))_{\alpha\in\Lambda}$ is convergent to 0 uniformly for $g \in \{f \in C_b(X, E) : \|f\|_p \leqslant 1\}$.*
6. *Assuming that $\mathbb{K} = \mathbb{R}$, the map $M : C_b(X) \to \mathbb{R}$ defined by $M(f) = \sup\{|T(g)| : g \in C_b(X, E)$ such that $\forall p \in s(E), \forall x \in X, p(g(x)) \leqslant f(x)\}$ for $f \geqslant 0$ and defined by $M(f) := M(f^+) - M(f^-)$ for $f \in C_b(X)$, is a linear functional $\mathcal{P}_p$-tight.*

Are satisfied the following statements:

(a) The propositions $1, 2$ and 3 are equivalents.

(b) The proposition 3 implies the proposition 4.

(c) *By assuming that* $\mathbb{K} = \mathbb{R}$ *and* T *is* τ_u*-continuous, we have:* $4 \Rightarrow 5 \Rightarrow 6$.

(d) *By assuming that* $\mathbb{K} = \mathbb{R}$ *and* E *is normed with norm* p*, we have:* $6 \Rightarrow 1$.

As a direct consequence from 3.1.6.a, we have the following:

Corollary 3.1.7. $(C_b(X, E), \beta_{\mathcal{P}})' = \bigcup_{p \in \mathcal{S}(E)} (C_b(X, E), \beta_p)'$.

In particular, we have the following result:

Corollary 3.1.8. *Suppose that* E *is a real vector space with norm* p*. Let* $T : C_b(X, E) \to \mathbb{R}$ *be a linear functional* τ_u*-continuous. The following propositions are equivalents:*

1. T *is* $\beta_{\mathcal{P}}$*-continuous.*

2. T *is* β_p*-continuous.*

3. T *is* $\mathcal{P}$*-tight.*

4. *Given* $\varepsilon > 0$*, there exists* $K \in \mathcal{P}$ *such that* $|T(f)| < \varepsilon$ *for each* $f \in C_b(X, E)$ *which satisfies* $\|f\|_p \leqslant 1$ *and* $f|_K = 0$.

5. *Any net* $(f_\alpha)_{\alpha \in \Lambda}$ *in* $\{f \in C_b(X) : \|f\| \leqslant 1\}$*, such that is* $\tau_{\mathcal{P}}$*-convergent to* 0*, satisfy the following condition:* $(T(f_\alpha g))_{\alpha \in \Lambda}$ *is convergent to* 0 *uniformly for* $g \in \{f \in C_b(X, E) : \|f\|_p \leqslant 1\}$.

6. *The map* $M : C_b(X) \to \mathbb{R}$ *defined by* $M(f) = \sup\{|T(g)| : g \in C_b(X, E)$ *such that* $\forall x \in X, p(g(x)) \leqslant f(x)\}$ *for* $f \geqslant 0$ *and by* $M(f) := M(f^+) - M(f^-)$ *for* $f \in C_b(X)$*, is a linear functional* $\mathcal{P}$*-tight.*

3.2. Integration in $C_b(X, E)$.

In this section, we will define the concept of integral for a function of the space $C_b(X, E)$, respect to a vector measure $m : \mathcal{B} \to E'$, which satisfy certain conditions.

This integral will be a generalization of the integral defined for functions in $C_b(X)$ respect to the Baire measures.

We will also develop the integration results that will be used in the following sections to study an integral representation of the $\beta_{\mathcal{P}}$-continuous functionals and operators on $C_b(X, E)$.

Throughout this section, we assume that $\mathbb{K} = \mathbb{R}$. For $A \in \mathcal{B}$, we will denote by $\Pi(A)$ the family of all the finite $\mathcal{B}$-partitions of A. Note that $\Pi(A)$ is a directed set with the relation $\preceq$ defined as follows: $C \preceq D$ if and only if D is a refinement of C.

Definition 3.2.1. *We denote by $S(X)$ the space of all real-simple functions with respect to $\mathcal{B}$. In other words:*

$$S(X) := \left\{ \sum_{i=1}^{n} a_i \mathscr{X}_{B_i} : n \in \mathbb{N}, \{B_i\}_{i=1}^{n} \in \Pi(X), a_i \in \mathbb{R} \right\} .$$

Given $f \in C_b(X)$, $f \geqslant 0$ and $\mu \in M^+(X)$, following [2], [1] and [12], it is defined the integral of f in X respect to μ, by:

$$\int_X f\, d\mu := \sup \left\{ \int_X h\, d\mu : h \in S(X), 0 \leqslant h \leqslant f \right\} .$$

It can be prove that:

$$\int_X f\, d\mu = \text{ínf} \left\{ \int_X h\, d\mu : h \in S(X), f \leqslant h \right\} = \lim_{n\to\infty} \int_X f_n\, d\mu = \lim_{n\to\infty} \int_X g_n\, d\mu \,,$$

where $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are arbitrary sequences of simple functions respect to $\mathcal{B}$, non-negatives, *pw*-convergent to f such that $f_n \leqslant f_{n+1} \leqslant f$ and $f \leqslant g_{n+1} \leqslant g_n$.

If $\mu \in M^+(X)$, $f \in C_b(X)$ and $f_1, f_2 \in C_b(X)$ are such that $f_1 \geqslant 0$, $f_2 \geqslant 0$ and $f = f_1 - f_2$, then, it is define the integral of f in X respect to μ, by:

$$\int_X f\, d\mu := \int_X f_1\, d\mu - \int_X f_2\, d\mu \,.$$

This integral is well defined, regardless of the choice of the functions f_1 and f_2.

Given $f \in C_b(X)$, $\mu \in M(X)$ and $\mu_1, \mu_2 \in M^+(X)$ such that $\mu = \mu_1 - \mu_2$, it is defined the integral of f in X respect to μ as:

$$\int_X f\, d\mu := \int_X f\, d\mu_1 - \int_X f\, d\mu_2 \,.$$

This integral is well defined, regardless of the choice of the measures μ_1 and μ_2.

In the following, we will give an equivalent definition for the integral of $f \in C_b(X)$ respect to a measure $\mu \in M(X)$. Then, this definition will be extended to elements of $C_b(X,E)$.

Definition 3.2.2. *Let $G \in \mathcal{B}$, $G \neq \emptyset$. It is defined the collection Ω_G as:*

$$\Omega_G := \left\{ \{B_1, \ldots, B_n, x_1, \ldots, x_n\} : n \in \mathbb{N}, \{B_i\}_{i=1}^n \in \Pi(G) \text{ and } x_i \in B_i \right\}.$$

This family is directed with the relation $\preceq$, defined as follows: given $\alpha_1 = \{B_1, \ldots, B_n, x_1, \ldots, x_n\}$ and $\alpha_2 = \{C_1, \ldots, C_m, y_1, \ldots, y_m\}$ in Ω_G, $\alpha_1 \preceq \alpha_2$ if and only if the partition $\{C_j\}_{j=1}^m$ is a refinement of $\{B_i\}_{i=1}^n$, this is, for each $j \in \{1, \ldots, m\}$, there exists $i \in \{1, \ldots, n\}$ such that $C_j \subset B_i$.

For each $f \in C_b(X)$, $\mu \in M(X)$ and $\alpha = \{B_1, \ldots, B_n, x_1, \ldots, x_n\} \in \Omega_X$ it is defined the integral

$$\omega_\alpha(f) := \omega_\alpha(f,\mu) := \sum_{i=1}^n f(x_i)\mu(B_i) = \int_X \sum_{i=1}^n f(x_i)\mathscr{X}_{B_i}\, d\mu.$$

Lemma 3.2.3. *Given $f \in C_b(X)$ and $\mu \in M(X)$ we have*

$$\int_X f\, d\mu = \lim_{\alpha \in \Omega_X} \omega_\alpha(f)\,.$$

Proof. It is enough to prove the lemma for $f \in C_b(X)$, $f \geq 0$ and $\mu \in M^+(X)$. For each $A \in \mathcal{B}$, we define $l(f,A) := \inf\{f(x) : x \in A\}$ and $u(f,A) := \sup\{f(x) : x \in A\}$. For each $\mathscr{C} = \{C_1, \ldots, C_n\} \in \Pi(X)$ we define $l(f,\mathscr{C}) := \sum_{i=1}^n l(f,C_i)\mu(C_i)$ and $u(f,\mathscr{C}) := \sum_{i=1}^n u(f,C_i)\mu(C_i)$.

<u>Step 1:</u> We will show that for $\mathscr{C},\mathscr{D} \in \Pi(X)$ such that $\mathscr{C} \preceq \mathscr{D}$, we have: $l(f,\mathscr{C}) \leqslant l(f,\mathscr{D})$ and $u(f,\mathscr{D}) \leqslant l(f,\mathscr{C})$.

Let $\mathscr{C},\mathscr{D} \in \Pi(X)$ be such that $\mathscr{C} = \{C_1, \ldots, C_n\} \preceq \mathscr{D} = \{D_1, \ldots, D_m\}$. There exists a partition $\{J_1, \ldots, J_n\}$ of set $\{1, \ldots, m\}$, such that:

$$C_i = \bigcup_{j \in J_i} D_j,$$

for each $i \in \{1, \dots, n\}$. Then,

$$\begin{aligned} l(f,\mathscr{C}) = \sum_{i=1}^{n} l(f,C_i)\mu(C_i) &= \sum_{i=1}^{n} l(f,C_i)\left(\sum_{j\in J_i}\mu(D_j)\right) \\ &= \sum_{i=1}^{n}\left(\sum_{j\in J_i} l(f,C_i)\mu(D_j)\right) \\ &\leqslant \sum_{i=1}^{n}\left(\sum_{j\in J_i} l(f,D_j)\mu(D_j)\right) = \sum_{j=1}^{m} l(f,D_j)\mu(D_j) = l(f,\mathscr{D})\,. \end{aligned}$$

Also, we have:

$$\begin{aligned} u(f,\mathscr{D}) = \sum_{j=1}^{m} u(f,D_i)\mu(D_j) &= \sum_{i=1}^{n}\left(\sum_{j\in J_i} u(f,D_j)\mu(D_j)\right) \\ &\leqslant \sum_{i=1}^{n}\left(\sum_{j\in J_i} u(f,C_i)\mu(D_j)\right) \\ &\leqslant \sum_{i=1}^{n} u(f,C_i)\left(\sum_{j\in J_i}\mu(D_j)\right) = \sum_{i=1}^{n} u(f,C_i)\mu(C_i) = u(f,\mathscr{C})\,. \end{aligned}$$

Noticing that for each $\mathscr{C} \in \Pi(X)$, $l(f,\mathscr{C}) \leqslant \|f\|\mu(X)$ and $-\|f\|\mu(X) \leqslant u(f,\mathscr{C})$. Therefore, $\text{ínf}\{u(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\}$ and $\sup\{l(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\}$ exist.

<u>Step 2:</u> We will show that $\int_X f\,d\mu = \text{ínf}\{u(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\} = \lim\limits_{\mathscr{C}\in\Pi(X)} u(f,\mathscr{C})\,.$

Let $g \in S(X)$ be such that $f \leq g$. Let us say $g = \sum_{i=1}^{n} \alpha_i \mathscr{X}_{D_i}$ and $\mathscr{D} = \{D_1, \dots, D_n\}$. We have:

$$\text{ínf}\{u(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\} \leq u(f,\mathscr{D}) = \sum_{i=1}^{n} u(f,D_i)\mu(D_i) \leq \sum_{i=1}^{n} \alpha_i\mu(D_i) = \int_X g\,d\mu\,.$$

Then,

$$\text{ínf}\{u(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\} \leq \text{ínf}\left\{\int_X g\,d\mu : g \in S(X), f \leq g\right\} = \int_X f\,d\mu\,.$$

Furthermore, if $\mathscr{C} = \{C_1, \dots, C_n\} \in \Pi(X)$, then, $f \leq \sum_{i=1}^{n} u(f,C_i)\mathscr{X}_{C_i}$. Then,

$$\int f\,d\mu \leq \sum_{i=1}^{n} u(f,C_i)\mu(C_i) = u(f,\mathscr{C}).$$

Therefore,

$$\int f\,d\mu \le \text{ínf}\{u(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\}.$$

Now consider $\varepsilon > 0$. There exists $\mathscr{C} \in \Pi(X)$ such that

$$\text{ínf}\{u(f,\mathscr{H}) : \mathscr{H} \in \Pi(X)\} + \varepsilon > u(f,\mathscr{C}).$$

Due to step 1, for $\mathscr{D} \in \Pi(X)$, $\mathscr{C} \preceq \mathscr{D}$, we have that $u(f,\mathscr{D}) \le u(f,\mathscr{C})$. Then, $|u(f,\mathscr{D}) - inf\{u(f,\mathscr{H}) : \mathscr{H} \in \Pi(X)\}| < \varepsilon$. Therefore, $\int_X f\,d\mu = \lim\limits_{\mathscr{C}\in\Pi(X)} u(f,\mathscr{C})$.

<u>STEP 3:</u> We will show that $\int_X f\,d\mu = \sup\{l(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\} = \lim\limits_{\mathscr{C}\in\Pi(X)} l(f,\mathscr{C})$.

Let $g \in S(X)$ be such that $g \le f$. Let us say $g = \sum_{i=1}^n \alpha_i \mathscr{X}_{D_i}$ y $\mathscr{D} = \{D_1,\dots,D_n\}$. We have:

$$\int_X g\,d\mu = \sum_{i=1}^n \alpha_i \mu(D_i) \le \sum_{i=1}^n l(f,D_i)\mu(D_i) = l(f,\mathscr{D}) \le \sup\{l(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\}.$$

Then,

$$\sup\{l(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\} \ge \sup\left\{\int_X g\,d\mu : g \in S(X), g \le f\right\} = \int_X f\,d\mu.$$

Furthermore, if $\mathscr{C} = \{C_1,\dots,C_n\} \in \Pi(X)$, then, $\sum_{i=1}^n l(f,C_i)\mathscr{X}_{C_i} \le f$. Then,

$$l(f,\mathscr{C}) = \sum_{i=1}^n l(f,C_i)\mu(C_i) \le \int f\,d\mu.$$

Therefore,

$$\sup\{l(f,\mathscr{C}) : \mathscr{C} \in \Pi(X)\} \le \int f\,d\mu.$$

Now consider $\varepsilon > 0$. There exists $\mathscr{C} \in \Pi(X)$, such that

$$\sup\{l(f,\mathscr{H}) : \mathscr{H} \in \Pi(X)\} - \varepsilon < l(f,\mathscr{C}).$$

Due to step 1, for $\mathscr{D} \in \Pi(X)$, $\mathscr{C} \preceq \mathscr{D}$, we have $l(f,\mathscr{C}) \le l(f,\mathscr{D})$. Then,

$$|sup\{l(f,\mathscr{H}) : \mathscr{H} \in \Pi(X)\} - l(f,\mathscr{D})| < \varepsilon.$$

Therefore, $\int_X f\,d\mu = \lim\limits_{\mathscr{C}\in\Pi(X)} l(f,\mathscr{C})$.

<u>STEP 4:</u> We will show that $\int_X f\,d\mu = \lim\limits_{\alpha\in\Omega_X} \omega_\alpha(f)$.

Let $\alpha = \{C_1,\dots,C_n,x_1,\dots,x_n\} \in \Omega_X$ and $\mathscr{C} = \{C_1,\dots,C_n\}$. We have

$$\sum_{i=1}^n l(f,C_i)\mu(C_i) \le \sum_{i=1}^n f(x_i)\mu(C_i) \le \sum_{i=1}^n u(f,C_i)\mu(C_i),$$

this is, $l(f,\mathscr{C}) \leq \omega_\alpha(f) \leq u(f,\mathscr{C})$. Then,

$$|\omega_\alpha(f)-l(f,\mathscr{C})| \leq |u(f,\mathscr{C})-l(f,\mathscr{C})| \leq \left|u(f,\mathscr{C})-\int_X f\,d\mu\right|+\left|\int_X f\,d\mu-l(f,\mathscr{C})\right|. \quad (3.1)$$

Due to steps 2 and 3, given $\varepsilon > 0$ there exists $\mathscr{H} = \{H_1,\dots,H_n\} \in \Pi(X)$, such that

$$\left|u(f,\mathscr{D}) - \int_X f\,d\mu\right| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \left|\int_X f\,d\mu - l(f,\mathscr{D})\right| < \frac{\varepsilon}{4}$$

for each $\mathscr{D} \in \Pi(X)$ such that $\mathscr{H} \preceq \mathscr{D}$.

For each $i = 1,\dots,n$, we choose $y_i \in H_i$ and write $\alpha_0 = \{H_1,\dots,H_n,y_1,\dots,y_n\} \in \Omega_X$. For $\alpha = \{D_1,\dots,D_m,x_1,\dots,x_m\} \in \Omega_X$ such that $\alpha_0 \preceq \alpha$, we have $\mathscr{D} = \{D_1,\dots,D_m\}$ is such that $\mathscr{H} \preceq \mathscr{D}$.

Thereby, applying (3.1) we obtain:

$$\begin{aligned}\left|\int_X f\,d\mu - \omega_\alpha(f)\right| &\leq \left|\int_X f\,d\mu - l(f,\mathscr{D})\right| + |l(f,\mathscr{D}) - \omega_\alpha(f)| \\ &\leq \left|\int_X f\,d\mu - l(f,\mathscr{D})\right| + \left|u(f,\mathscr{D}) - \int_X f\,d\mu\right| + \left|\int_X f\,d\mu - l(f,\mathscr{D})\right| \\ &< \varepsilon\,.\end{aligned}$$

□

Given $p \in s(E)$ we will denote by E'_p the dual space $(E,\sigma(E,\{p\}))'$.

Definition 3.2.4. *For each $p \in s(E)$ it is defined the space $M_p(X,E'_p)$, as the set of all finitely additive set functions $m : \mathcal{B} \to E'_p$ that satisfies the following conditions:*

(a) For each $s \in E$, the set function $ms : \mathcal{B} \to \mathbb{R}$, defined by $ms(G) := \langle m(G), s\rangle$, is a Baire measure. That is to say, $\forall s \in E$, $ms \in M(X)$.

(b) $m_p(X) < \infty$, where for each $G \in \mathcal{B}$, it is define

$$m_p(G) := \sup\left\{\left|\sum_{i=1}^n \langle m(G_i), s_i\rangle\right| : \begin{array}{l} n \in \mathbb{N}, \{G_i\}_{i=1}^n \in \Pi(G), s_i \in E, \\ p(s_i) \leq 1, i = 1,\dots,n \end{array}\right\}$$

Remark 3.2.5. *If a set function $m : \mathcal{B} \to E'$ satisfies 3.2.4.a then is a vectorial measure, that is to say, $m(\emptyset) = 0$ and $m(A \cup B) = m(A) + m(B)$ for disjoint $A, B \in \mathcal{B}$. In fact, for each $s \in E$, $\langle m(\emptyset), s\rangle = ms(\emptyset) = 0$, since $ms \in M(X)$. Now consider disjoint $A, B \in \mathcal{B}$. For each $s \in E$, $\langle m(A \cup B), s\rangle = ms(A \cup B) = ms(A) + ms(B) = \langle m(A), s\rangle + \langle m(B), s\rangle$.*

Remark 3.2.6. *Let $p \in s(E)$. If a set function $m : \mathcal{B} \to E'$ satisfies 3.2.4.b, then, the set function m_p is bounded and the range of the function m is contained in E'_p.*

In fact, given $G \in \mathcal{B}$, $\{G_i\}_{i=1}^n \in \Pi(G)$ and $s_i \in E$, $p(s_i) \leq 1$ $(i = 1, \ldots, n)$, we have $\left|\sum_{i=1}^n \langle m(G_i), s_i\rangle\right| = \left|\sum_{i=1}^n \langle m(G_i), s_i\rangle + \langle m(X\backslash G), 0\rangle\right| \leq m_p(X)$. Therefore, $\|m(G)\|_{p,\infty} := \sup\{|\langle m(G), s\rangle| : p(s) \leq 1\} \leq m_p(G) \leq m_p(X) < \infty$,for each $G \in \mathcal{B}$.

Remark 3.2.7. *Let $p \in s(E)$, $m \in M_p(X, E'_p)$ and $G \in \mathcal{B}$. Note that:*

$$m_p(G) = \sup\left\{\sum_{i=1}^n \left|\langle m(G_i), s_i\rangle\right| : \begin{array}{l} n \in \mathbb{N}, \{G_i\}_{i=1}^n \in \Pi(G), s_i \in E, \\ p(s_i) \leq 1, i = 1, \ldots, n \end{array}\right\}.$$

In fact, is clear that

$$m_p(G) \leq \sup\left\{\sum_{i=1}^n \left|\langle m(G_i), s_i\rangle\right| : \begin{array}{l} n \in \mathbb{N}, \{G_i\}_{i=1}^n \in \Pi(G), s_i \in E, \\ p(s_i) \leq 1, i = 1, \ldots, n \end{array}\right\}.$$

For the other inequality, consider $\{G_i\}_{i=1}^n \in \Pi(G)$ and $s_i \in E$, $p(s_i) \leq 1$. For each $i \in \{1, \ldots, n\}$, we define

$$t_i = \begin{cases} s_i & , \textit{ if } \ \langle m(G_i), s_i\rangle \geq 0 \\ -s_i & , \textit{ if } \ \langle m(G_i), s_i\rangle < 0 \end{cases}$$

Then,

$$\sum_{i=1}^n \left|\langle m(G_i), s_i\rangle\right| = \sum_{i=1}^n \langle m(G_i), t_i\rangle = \left|\sum_{i=1}^n \langle m(G_i), t_i\rangle\right| \leq m_p(G)\,.$$

Therefore,

$$\sup\left\{\sum_{i=1}^n \left|\langle m(G_i), e_i\rangle\right| : \begin{array}{l} n \in \mathbb{N}, \{G_i\}_{i=1}^n \in \Pi(G), e_i \in E, \\ p(e_i) \leq 1, i = 1, \ldots, n \end{array}\right\} \leq m_p(G)\,.$$

The other inequality is obtained by applying the triangle inequality.

Lemma 3.2.8. *If $p \in s(E)$ and $m \in M_p(X, E'_p)$, then the set function $m_p : \mathcal{B} \to \mathbb{R}$, defined in 3.2.4.b, is a positive Baire measure, this is $m_p \in M^+(X)$.*

Proof. Let $A, B \in \mathcal{B}$ be disjoint. We can assume that $A \neq \emptyset$ and $B \neq \emptyset$. Let $\{C_i\}_{i=1}^n \in \Pi(A \cup B)$ and $s_i \in E$, $p(s_i) \leq 1$. For each $i \in \{1, \ldots, n\}$, we consider the sets

$$A_i := A \cap C_i \quad \text{and} \quad B_i := B \cap C_i$$

and write $I = \{i \in \{1, \ldots, n\} : A_i \neq \emptyset\}$ and $J = \{j \in \{1, \ldots, n\} : B_j \neq \emptyset\}$. By construction, we have that $\{A_i\}_{i \in I} \in \Pi(A)$, $\{B_j\}_{j \in J} \in \Pi(B)$ and $C_i = C_i \cap (A \cup B) = A_i \cup B_i$. Then, by the remark 3.2.7,

$$\begin{aligned}
\sum_{i=1}^{n} \big|\langle m(C_i), s_i \rangle\big| &= \sum_{i=1}^{n} \big|\langle m(A_i \cup B_i), s_i \rangle\big| \\
&= \sum_{i=1}^{n} \big|\langle m(A_i), s_i \rangle + \langle m(B_j), s_j \rangle\big| \\
&\leq \sum_{i \in I} \big|\langle m(A_i), s_i \rangle\big| + \sum_{j \in J} \big|\langle m(B_j), s_j \rangle\big| \\
&\leq m_p(A) + m_p(B).
\end{aligned}$$

Therefore $m_p(A \cup B) \leq m_p(A) + m_p(B)$.

Let $\varepsilon > 0$. Consider $\{A_i\}_{i=1}^n \in \Pi(A)$, $s_i \in E$, $p(s_i) \leq 1$, $\{B_j\}_{j=1}^m \in \Pi(B)$, $e_j \in E$, $p(e_j) \leq 1$ such that:

$$m_p(A) < \sum_{i=1}^{n} \big|\langle m(A_i), s_i \rangle\big| + \frac{\varepsilon}{2} \quad \text{and} \quad m_p(B) < \sum_{j=1}^{m} \big|\langle m(B_i), e_j \rangle\big| + \frac{\varepsilon}{2}.$$

Then,

$$m_p(A) + m_p(B) < \varepsilon + \sum_{i=1}^{n} \big|\langle m(A_i), s_i \rangle\big| + \sum_{j=1}^{m} \big|\langle m(B_i), e_j \rangle\big| \leq \varepsilon + m_p(A \cup B)$$

Let $A \in \mathcal{B}$. Since m_p is non-negative and finitely additive, applying 1.3.1.7, we have that m_p is monotone. Therefore, we have $\sup\{m_p(Z) : Z \in \mathcal{Z}, Z \subset A\} \leq m_p(A)$.

For the other inequality, consider $\varepsilon > 0$. There exist $\{A_i\}_{i=1}^n \in \Pi(A)$, $s_i \in E$, $p(s_i) \leq 1$, for $i = 1, \ldots, n$, such that

$$m_p(A) < \frac{\varepsilon}{2} + \sum_{i=1}^{n} \left| ms_i(A_i) \right| = \frac{\varepsilon}{2} + \sum_{i=1}^{n} mt_i(A_i),$$

where for each $i \in \{1, \ldots, n\}$,

$$t_i = \begin{cases} s_i & , \text{ if } \ \langle m(G_i), s_i \rangle \geq 0 \\ -s_i & , \text{ if } \ \langle m(G_i), s_i \rangle < 0 \ . \end{cases}$$

By the regularity of the measures $(mt_i)^+$, we can choose for each $i \in \{1 \ldots, n\}$, a set $Z_i \subset A_i$, such that

$$(mt_i)^+(A_i) - \frac{\varepsilon}{2n} = \sup\{(mt_i)^+(Z) : Z \in \mathcal{Z}, Z \subset A_i\} - \frac{\varepsilon}{2n} < (mt_i)^+(Z_i).$$

Note that $(mt_i)^-(Z_i) \leq (mt_i)^-(A_i)$. Then,

$$\begin{aligned} m_p(A) &< \varepsilon - \frac{\varepsilon}{2} + \sum_{i=1}^{n} (mt_i)^+(A_i) - \sum_{i=1}^{n} (mt_i)^-(A_i) \\ &< \varepsilon + \sum_{i=1}^{n} (mt_i)^+(Z_i) - \sum_{i=1}^{n} (mt_i)^-(Z_i) \\ &= \varepsilon + \sum_{i=1}^{n} (mt_i)(Z_i) \\ &\leq \varepsilon + m_p\left(\bigcup_{i=1}^{n} Z_i\right) \\ &\leq \varepsilon + \sup\{m_p(Z) : Z \in \mathcal{Z}, Z \subset A\} \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that: $m_p(A) = \sup\{m_p(Z) : Z \in \mathcal{Z}, Z \subset A\}$ and therefore $m_p \in M^+(X)$. □

Definition 3.2.9. *A measure $\mu \in M(X)$ is **$\mathcal{P}$-tight** if and only if for each $\varepsilon > 0$, there exists $K \in \mathcal{P}$ such that $|\mu|(G) < \varepsilon$, for each $G \in \mathcal{B}$, $G \subset X \backslash K$.*

*If $\mathcal{P} = K(X)$, then, to say that a Baire measure μ is $\mathcal{P}$-tight, we say μ is **tight**.*

It is define the space $M_{\mathcal{P},t}(X) := \{\mu \in M(X) : \mu \text{ is } \mathcal{P}\text{-tight}\}$ and we will denote by $M_t(X)$ the space $M_{K(X),t}$.

Note that $M_{\mathcal{P},t}(X) \subset M_t(X)$, since $\mathcal{P} \subset K(X)$.

Lemma 3.2.10. *Let $\mu \in M(X)$. The following statements are equivalents:*

(a) The measure μ is $\mathcal{P}$-tight.

(b) For each $\varepsilon > 0$, there exists $K \in \mathcal{P}$ such that $|\mu|(Z) < \varepsilon$ for each $Z \in \mathcal{Z}$, $Z \subset X \backslash K$.

Proof. $(a) \Rightarrow (b)$: Is immediate. $(b) \Rightarrow (a)$: Let $\varepsilon > 0$. There exists $K \in \mathcal{P}$ such that $|\mu|(Z) < \frac{\varepsilon}{2}$ for each $Z \in \mathcal{Z}$, $Z \subset X\backslash K$. We take $G \in \mathcal{B}$, $G \subset X\backslash K$. Since $|\mu|(Z) < \frac{\varepsilon}{2}$, for each $Z \in \mathcal{Z}$, $Z \subset G$ and by the regularity of μ, we have $|\mu|(G) = \sup\{|\mu|(Z) : Z \in \mathcal{Z}, Z \subset G\} < \varepsilon$. □

Lemma 3.2.11. *Let $\mu \in M(X)$, $\varepsilon > 0$ and $K \in K(X)$. The following statements are equivalents:*

(a) $\sup\{|\mu|(Z) : Z \in \mathcal{Z}, Z \subset X\backslash K\} \le \varepsilon$,

(b) For each $U \in \mathcal{U}$ such that $K \subset U$, we have $|\mu|(X) \le |\mu|(U) + \varepsilon$.

Proof. $(a) \Rightarrow (b)$: Let $U \in \mathcal{U}$ such that $K \subset U$. Then, $X\backslash U \subset X\backslash K$ and by (a) we have $|\mu|(X\backslash U) \le \varepsilon$. Thereby, $|\mu|(X) = |\mu|(U) + |\mu|(X\backslash U) \le |\mu|(U) + \varepsilon$, concluding (b).

$(b) \Rightarrow (a)$: If $Z \in \mathcal{Z}$ is such that $Z \subset X\backslash K$, then, by (b) we have $|\mu|(X) = |\mu|(Z) + |\mu|(X\backslash Z) \le |\mu|(X\backslash Z) + \varepsilon$ and from this, it is concluded that $|\mu|(Z) \le \varepsilon$.

By the arbitrariness of Z, it follows (a). □

Lemma 3.2.12. *Let $p \in s(E)$ and $m \in M_p(X, E'_p)$. If for each $s \in E$, $ms \in M_t(X)$, then $m_p \in M_t(X)$.*

Proof. Let $\varepsilon > 0$. By the remark 3.2.7 and by a similar argument used in 3.2.8, there exists a collection $\{Z_i\}_{i=1}^n \subset \mathcal{Z}$ such that $Z_i \cap Z_j = \emptyset$ for $i \neq j$ and a collection $\{t_i\}_{i=1}^n \subset E$ such that $p(t_i) \le 1$ and

$$m_p(X) < \frac{\varepsilon}{3} + \sum_{i=1}^{n} |mt_i(Z_i)|.$$

According to the previous lemma and the hypothesis, for each $i \in \{1, \ldots, n\}$, we can choose a set $H_i \in \mathcal{P}$ such that $|mt_i|(X) < \frac{\varepsilon}{6n} + |mt_i|(U)$ for each $U \in \mathcal{U}$, $H_i \subset U$.

For each $i \in \{1, \ldots, n\}$, we define $K_i := H_i \cap Z_i$. If $W \in \mathcal{U}$ is such that $K_i \subset W$, then $H_i \subset W \cup (X \backslash Z_i) \in \mathcal{U}$, and from this it follows that:

$$\begin{aligned} |mt_i|(Z_i) + |mt_i|(X \backslash Z_i) = |mt_i|(X) &< \frac{\varepsilon}{6n} + |mt_i|(W \cup (X \backslash Z_i)) \\ &\leq \frac{\varepsilon}{6n} + |mt_i|(W) + |mt_i|(X \backslash Z_i). \end{aligned}$$

Then, $|mt_i|(Z_i) < \frac{\varepsilon}{6n} + |mt_i|(W)$, for each $i \in \{1, \ldots, n\}$.

By the regularity of each $|mt_i|$ and by the result 1.3.3.3, there exists $U_i \in \mathcal{U}$ such that: $Z_i \subset U_i$ and $|mt_i|(U_i) < \frac{\varepsilon}{6n} + |mt_i|(Z_i)$, for each $i = 1, \ldots, n$. Applying the lemma 1.3.2.7, we can choose a collection $\{O_i\}_{i=1}^{n} \subset \mathcal{U}$ such that $Z_i \subset O_i \subset U_i$ and $O_i \cap O_j = \emptyset$ for $i \neq j$. From this, we obtain that: $|mt_i|(O_i) \leq |mt_i|(U_i) < \frac{\varepsilon}{6n} + |mt_i|(Z_i)$, for each $i = 1, \ldots, n$.

Let $K = \bigcup_{i=1}^{n} K_i \in K(X)$ and $W \in \mathcal{U}$ such that $K \subset W$. For each $i \in \{1, \ldots, n\}$, let us take $V_i := O_i \cap W \in \mathcal{U}$ and define $V := \bigcup_{i=1}^{n} V_i$. Since $K_i \subset V_i \subset O_i$, we have: $|mt_i|(V_i) \leq |mt_i|(O_i) \leq |mt_i|(Z_i) + \frac{\varepsilon}{6n} \leq |mt_i|(V_i) + \frac{\varepsilon}{3n}$, for $i \in \{1, \ldots, n\}$.

Then, $|mt_i(O_i \backslash V_i)| \leq |mt_i|(O_i \backslash V_i) \leq \frac{\varepsilon}{3n}$ and thereby: $|mt_i(O_i)| \leq |mt_i(V_i)| + \frac{\varepsilon}{3n}$, for each $i \in \{1, \ldots, n\}$.

Since $|mt_i|(O_i \backslash Z_i) \leq \frac{\varepsilon}{6n}$, we have $|mt_i(O_i) - mt_i(Z_i)| = |mt_i(O_i \backslash Z_i)| \leq |mt_i|(O_i \backslash Z_i) < \frac{\varepsilon}{3n}$, and from this, it is concluded that $|mt_i(Z_i)| < \frac{\varepsilon}{3n} + |mt_i(O_i)|$, for $i = 1, \ldots, n$.

Finally, we have:

$$\begin{aligned} m_p(X) < \frac{\varepsilon}{3} + \sum_{i=1}^{n} |mt_i(Z_i)| &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{n} |mt_i(O_i)| \\ &\leq \varepsilon + \sum_{i=1}^{n} |mt_i(V_i)| \\ &\leq \varepsilon + m_p(V) \\ &\leq \varepsilon + m_p(W). \end{aligned}$$

We conclude the proof by applying the lemmas 3.2.11 and 3.2.10. □

Lemma 3.2.13. *Let $p \in s(E)$ and $m \in M_p(X,E'_p)$. If $m_p \in M_{\mathcal{P},t}(X)$, then, $ms \in M_{\mathcal{P},t}(X)$, for each $s \in E$.*

Proof. Let $s \in E$. If $p(s) \leq 1$, then, $ms(G) \leq m_p(G)$, for all $G \in \mathcal{B}$. On the other hand, if $p(s) \geq 1$, then, $\frac{1}{p(s)} ms(G) \leq m_p(G)$, for all $G \in \mathcal{B}$.

Then, in any of the previous cases, if $m_p \in M_{\mathcal{P},t}(X)$, then, $ms \in M_{\mathcal{P},t}(X)$. □

Definition 3.2.14. *For each $p \in s(E)$, we define the space*

$$M_{\mathcal{P},p,t}(X,E'_p) := \{m \in M_p(X,E'_p) : m_p \in M_{\mathcal{P},t}(X)\}.$$

Additionally, the following spaces are define:

$$M(X,E') := \bigcup_{p\in s(E)} M_p(X,E'_p)$$

$$M_{\mathcal{P},t}(X,E') := \bigcup_{p\in s(E)} M_{\mathcal{P},p,t}(X,E'_p)\,.$$

Let $p \in s(E)$, $m \in M_{\mathcal{P},p,t}(X,E'_p)$, $G \in \mathcal{B}$, $G \neq \emptyset$ and $f : X \to E$ be an arbitrary function. For $\alpha = \{G_1,\ldots,G_n,x_1,\ldots,x_n\} \in \Omega_G$, we write

$$\omega_\alpha(f) := \omega_\alpha(f,m) := \sum_{i=1}^{n} \langle m(G_i), f(x_i)\rangle.$$

Definition 3.2.15. *Let $p \in s(E)$, $m \in M_{\mathcal{P},p,t}(X,E'_p)$, $G \in \mathcal{B}$, $G \neq \emptyset$ and f be an arbitrary function of X in E. We will say f is **m-integrable in G** if there exists $\lim_{\alpha\in\Omega_G} \omega_\alpha(f)$.*

*In this case it is define the **integral of f in G respect to m**, by*

$$\int_G f\,dm := \lim_{\alpha\in\Omega_G} \omega_\alpha(f).$$

Also, we will denote $\int_\emptyset f\,dm := 0$.

Lemma 3.2.16. *Let $p \in s(E)$, $m \in M_{\mathcal{P},p,t}(X,E'_p)$, $G \in \mathcal{B}$, $G \neq \emptyset$ and $f \in C_b(X,E)$. The net $(\omega_\alpha(f))_{\alpha \in \Omega_G}$ is Cauchy in $\mathbb{R}$ and therefore is convergent.*

Proof. Let $\varepsilon > 0$ and $r > 0$ be such that $\|f\|_p < r$. Taking $K \in \mathcal{P}$ such that $m_p(H) \leq \frac{\varepsilon}{r}$, for each $H \in \mathcal{B}$, $H \subset X \backslash K$. Since $f(K)$ is compact, there exist $x_1, \dots, x_n \in K$ such that:

$$f(K) \subset \bigcup_{i=1}^{n} \{e \in E : p(e - f(x_i)) \leq \varepsilon\}.$$

For each $i = 1, \dots, n$,

$$V_i := f^{-1}(\{e \in E : p(e - f(x_i)) \leq \varepsilon\}) = \{x \in X : p(f(x) - f(x_i)) \leq \varepsilon\} \in \mathcal{Z}.$$

Thereby, $K \subset \bigcup_{i=1}^{n} V_i$. For each $i = 1, \dots, n$, we define $G'_i := V_i \cap G$.

<u>Case 1:</u> There exists $i \in \{1, \dots, n\}$ such that $V_i \cap G \neq \emptyset$.

By defining $G_1 = G'_1$, $G_i = G'_i \backslash \left(\bigcup_{j=1}^{i-1} G_i \right)$, for $i = 2, \dots, n$, and keeping those $G_i \neq \emptyset$, we obtain a collection of disjoint Baire sets of pairs, $\{G_1, \dots, G_k\}$. We define $G_{k+1} := G \backslash \left(\bigcup_{j=1}^{k} G_j \right)$.

<u>Case 1.1:</u> $G_{k+1} \neq \emptyset$.

The collection $\{G_1, \dots, G_{k+1}\}$, a $\mathcal{B}$-partition of G, such that $G_{k+1} \subset X \backslash K$ and $p(f(x) - f(y)) \leq 2\varepsilon$, for $x, y \in G_i$, with $i = 1, \dots, k$. Take $y_i \in G_i$, for each $i \in \{1, \dots, k+1\}$ and consider $\alpha_0 := \{G_1, \dots, G_{k+1}, y_1, \dots, y_{k+1}\} \in \Omega_G$.

Given $\alpha_1, \alpha_2 \in \Omega_G$ such that $\alpha_0 \preceq \alpha_1, \alpha_2$, we have

$$|\omega_{\alpha_1}(f) - \omega_{\alpha_2}(f)| \leq |\omega_{\alpha_1}(f) - \omega_{\alpha_0}(f)| + |\omega_{\alpha_0}(f) - \omega_{\alpha_2}(f)|.$$

If $\alpha_1 = \{A_1, \dots, A_n, x_1, \dots, x_n\}$, then, there exists a partition $\{J_1, \dots, J_{k+1}\}$ of the set $\{1, \dots, n\}$, such that:

$$G_i = \bigcup_{j \in J_i} A_j,$$

for each $i \in \{1, \dots, k+1\}$.

Thereby,

$$
\begin{aligned}
|\omega_{\alpha_1}(f) - \omega_{\alpha_0}(f)| &\leq \left| \sum_{j=1}^{n} \langle m(A_j), f(x_j)\rangle - \sum_{i=1}^{k+1} \langle m(G_i), f(y_i)\rangle \right| \\
&\leq \sum_{i=1}^{k} \left| \sum_{j\in J_i} \langle m(A_j), f(x_j)\rangle - \langle m(G_i), f(y_i)\rangle \right| \\
&\qquad\qquad + \left| \sum_{j\in J_{k+1}} \langle m(A_j), f(x_j)\rangle - \langle m(G_{k+1}), f(y_{k+1})\rangle \right| \\
&= \sum_{i=1}^{k} \left| \sum_{j\in J_i} \langle m(A_j), f(x_j)\rangle - \sum_{j\in J_i} \langle m(A_j), f(y_i)\rangle \right| \\
&\qquad\qquad + \left| \sum_{j\in J_{k+1}} \langle m(A_j), f(x_j)\rangle - \sum_{j\in J_{k+1}} \langle m(A_j), f(y_{k+1})\rangle \right| \\
&= 2\varepsilon \sum_{i=1}^{k} \left| \sum_{j\in J_i} \left\langle m(A_j), \frac{f(x_j) - f(y_i)}{2\varepsilon} \right\rangle \right| \\
&\qquad\qquad + 2r \left| \sum_{j\in J_{k+1}} \left\langle m(A_j), \frac{f(x_j) - f(y_{k+1})}{2r} \right\rangle \right| \\
&\leq 2\varepsilon \sum_{i=1}^{k} m_p(G_i) + 2r \cdot m_p(G_{k+1}) \\
&\leq 2\varepsilon \sum_{i=1}^{k} m_p(G_i) + 2\varepsilon \\
&= 2\varepsilon \left(m_p \left(\bigcup_{i=1}^{k} G_i \right) + 1 \right) \\
&\leq 2\varepsilon (m_p(X) + 1).
\end{aligned}
$$

Analogously, $|\omega_{\alpha_0}(f) - \omega_{\alpha_2}(f)| \leq 2\varepsilon(m_p(X) + 1)$. Then, $|\omega_{\alpha_1}(f) - \omega_{\alpha_2}(f)| \leq 4\varepsilon(m_p(X) + 1)$.

<u>Case 1.2:</u> $G_{k+1} = \emptyset$, this is, $G = \bigcup_{i=1}^{k} G_k$.

In this case, the collection $\{G_1, \ldots, G_k\}$ is a $\mathcal{B}$-partition of G and for each $i \in \{1, \ldots, k\}$ we consider $y_i \in G_i$. We define $\alpha_0 = \{G_1, \ldots, G_k, y_1, \ldots, y_k\} \in \Omega_G$ and proceed analogously to case 1.1, concluding that for $\alpha_1, \alpha_2 \in \Omega_G$ such that $\alpha_0 \preceq \alpha_1, \alpha_2$, we have $|\omega_{\alpha_1}(f) - \omega_{\alpha_2}(f)| \leq 4\varepsilon(m_p(X) + 1)$.

<u>Case 2:</u> For each $i \in \{1,\ldots,n\}$, $V_i \cap G = \emptyset$.

In this case, $K \cap G \subset (\bigcup_{i=1}^{n} V_i) \cap G = \emptyset$, therefore, $G \subset X \backslash K$. Then, $m_p(G) \leq \frac{\varepsilon}{r}$. We choose $x \in G$ and define $\alpha_0 = \{G, x\} \in \Omega_G$. For $\alpha_1 = \{G_i,\ldots,G_n,x_1,\ldots,x_n\} \in \Omega_G$, such that $\alpha_0 \preceq \alpha_1$, we have

$$\begin{aligned}
|\omega_{\alpha_0}(f) - \omega_{\alpha_1}(f)| &= |\langle m(G), f(x)\rangle - \sum_{i=1}^{n} \langle m(G_i), f(x_i)\rangle| \\
&= \left| \sum_{i=1}^{n} \langle m(G_i), f(x)\rangle - \sum_{i=1}^{n} \langle m(G_i), f(x_i)\rangle \right| \\
&= 2r \sum_{i=1}^{n} \left| \left\langle m(G_i), \frac{f(x)-f(x_i)}{2r} \right\rangle \right| \\
&\leq 2r \cdot m_p(G) \\
&\leq 2\varepsilon \\
&\leq 2\varepsilon(m_p(X)+1).
\end{aligned}$$

Thereby, we have proved that given $\varepsilon > 0$, there exists $\alpha_0 \in \Omega_G$, such that for $\alpha_1, \alpha_2 \in \Omega_G$ such that $\alpha_0 \preceq \alpha_1, \alpha_2$, we have $|\omega_{\alpha_1}(f) - \omega_{\alpha_2}(f)| \leq 4\varepsilon(m_p(X)+1)$. □

Proposition 3.2.17. *Let $p \in s(E)$, $m \in M_{\mathcal{P},p,t}(X, E'_p)$, $G, G_1, G_2 \in \mathcal{B}$ and $f \in C_b(X,E)$. Are satisfy the following statements:*

(a) *f is m-integrable in G.*

(b) *If G_1, G_2 are disjoint, then* $\displaystyle\int_{G_1 \cup G_2} f\,dm = \int_{G_1} f\,dm + \int_{G_2} f\,dm.$

(c) $\displaystyle\left|\int_G f\,dm\right| \leq m_p(G) \sup_{x\in G}\{p(f(x))\}$

Proof. The statement (a) is consequence from the previous result.

Let us prove (b). The case in which G_1 or G_2 is the empty set, is immediate. Suppose that G_1 and G_2 are different from the empty set. Given $\varepsilon > 0$, there exist $\alpha_{1,0} = \{V_1,\ldots,V_n,x_1,\ldots,x_n\} \in \Omega_{G_1}, \alpha_{2,0} = \{W_1,\ldots,W_m,y_1,\ldots,y_m\} \in \Omega_{G_2}$, such that:

$$\left|\int_{G_1} f\,dm - \omega_{\alpha_1}(f)\right| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \left|\int_{G_2} f\,dm - \omega_{\alpha_2}(f)\right| < \frac{\varepsilon}{2},$$

for $\alpha_1 \in \Omega_{G_1}, \alpha_2 \in \Omega_{G_2}$ such that $\alpha_{1,0} \preceq \alpha_1$ y $\alpha_{2,0} \preceq \alpha_2$.

Consider

$$\alpha_{1,0} \cup \alpha_{2,0} := \{V_1, \dots, V_n, W_1, \dots, W_m, x_1, \dots, x_n, y_1, \dots, y_m\} \in \Omega_{G_1 \cup G_2}.$$

If $\alpha := \{S_1, \dots, S_k, s_1, \dots, y_m\} \in \Omega_{G_1 \cup G_2}$ is such that $\alpha_{1,0} \cup \alpha_{2,0} \preceq \alpha$, then, we can make a decomposition of the form $\alpha = \alpha_1' \cup \alpha_2'$, where $\alpha_1' \in \Omega_{G_1}$, $\alpha_2' \in \Omega_{G_2}$ such that $\alpha_{1,0} \preceq \alpha_1'$ and $\alpha_{2,0} \preceq \alpha_2'$.

Then, we have that:

$$\begin{aligned}\left|\int_{G_1} f\,dm + \int_{G_2} f\,dm - \omega_\alpha(f)\right| &= \left|\int_{G_1} f\,dm + \int_{G_2} f\,dm - (\omega_{\alpha_1'}(f) + \omega_{\alpha_2'}(f))\right| \\ &\le \left|\int_{G_1} f\,dm - \omega_{\alpha_1'}(f)\right| + \left|\int_{G_2} f\,dm - \omega_{\alpha_2'}(f)\right| \\ &\le \varepsilon.\end{aligned}$$

Since the net $(\omega_\alpha(f))_{\alpha \in \Omega_{G_1 \cup G_2}}$ converges to an only value, it is concluded that

$$\int_{G_1 \cup G_2} f\,dm = \lim_{\alpha \in \Omega_{G_1 \cup G_2}} \omega_\alpha(f) = \int_{G_1} f\,dm + \int_{G_2} f\,dm\,.$$

Let us prove (c): Let $\alpha = \{G_1, \dots, G_n, x_1, \dots, x_n\} \in \Omega_G$. We have that:

$$\left|\left|\int_G f\,dm\right| - \left|\omega_\alpha(f)\right|\right| \le \left|\int_G f\,dm - \omega_\alpha(f)\right|.$$

Therefore, $\lim_{\alpha \in \Omega_G} |\omega_\alpha(f)| = \left|\int_G f\,dm\right|$. Putting $s = \sup\{p(f(x)) : x \in G\}$.

If $s = 0$, then we have that:

$$|\omega_\alpha(f)| \le \sum_{i=1}^n |\langle m(G_i), f(x_i)\rangle| \le \sum_{i=1}^n \|m(G_i)\| p(f(x_i)) \le \sum_{i=1}^n \|m(G_i)\| s = 0 = m_p(G) \cdot s,$$

where $\|m(G_i)\| = \sup\{|\langle m(G_i), s\rangle| : p(s) \le 1\} < \infty$, since $m(G_i) \in E_p'$.

If $s \neq 0$, then

$$|\omega_\alpha(f)| \le \sum_{i=1}^n |\langle m(G_i), f(x_i)\rangle| = s \sum_{i=1}^n \left|\left\langle m(G_i), \frac{f(x_i)}{s}\right\rangle\right| \le s \cdot m_p(G).$$

By the arbitrariness of α, in both cases, it follows that $|\int_G f\,dm| \le s \cdot m_p(G)$. □

Remark 3.2.18. *Note that if $E = \mathbb{R}$, then, due to the lemma 3.2.3, the defined integral in 3.2.15 coincide with the defined in [2], [1] and [12].*

Proposition 3.2.19. *Let $\varepsilon > 0$, $K \in K(X)$ and $p \in s(E)$. Putting $W = \{f \in C_b(X,E) : \|f\|_p \leq 1$ and $f|_K = 0\}$. For $m \in M_p(X, E'_p)$ such that $m_p \in M_t(X)$, the following statements are equivalents:*

(a) $m_p(G) \leq \varepsilon$, *for each* $G \in \mathcal{B}$ *such that* $G \subset X \backslash K$.

(b) $\left| \int_X f \, dm \right| \leq \varepsilon$, *for each* $f \in W$.

Proof. $(a) \Rightarrow (b)$: Let $\varepsilon_1 > 0$ and $f \in W$. Considering the set $Z := \{x \in X : p(f(x)) \geq \varepsilon_1\}$. We have that $Z \in \mathcal{Z}$ and $Z \subset X \backslash K$. Thereby, applying the previous proposition we obtain:

$$\left| \int_X f \, dm \right| \leq \left| \int_Z f \, dm \right| + \left| \int_{X \backslash Z} f \, dm \right| \leq m_p(Z) + \varepsilon_1 \cdot m_p(X \backslash Z) \leq \varepsilon + \varepsilon_1 \cdot m_p(X).$$

Since $\varepsilon_1 > 0$ is arbitrary, we have that $\left| \int_X f \, dm \right| \leq \varepsilon$.

$(b) \Rightarrow (a)$: By the regularity of m_p, it is enought to prove that: $m_p(Z) \leq \varepsilon$, for each $Z \in \mathcal{Z}$ such that $Z \subset X \backslash K$. Let $Z \in \mathcal{Z}$ such that $Z \subset X \backslash K$. There exists $h \in C_b(X)$ such that $0 \leq h \leq \mathscr{X}_X$, $h(Z) = \{0\}$ and $h(K) = \{1\}$. If we write $U = \{x \in X : h(x) < \frac{1}{2}\}$, then $U \in \mathcal{U}$ and $Z \subset U \subset X \backslash K$. For $\varepsilon_1 > 0$, there exists $\{G_1, \ldots, G_n\} \in \Pi(Z)$ and $\{s_i\}_{i=1}^n \subset E$ such that $p(s_i) \leq 1$ and

$$m_p(Z) < \sum_{i=1}^n ms_i(G_i) + \varepsilon_1 - \frac{\varepsilon_1}{2}.$$

By the regularity of the measures ms_i^+ $(i = 1, \ldots, n)$, there exist $Z_1, \ldots, Z_n \in \mathcal{Z}$ such that $Z_i \subset G_i$ and

$$\begin{aligned} ms_i(G_i) - \frac{\varepsilon_1}{2n} &= ms_i^+(G_i) - ms_i^-(G_i) - \frac{\varepsilon_1}{2n} \\ &\leq ms_i^+(Z_i) - ms_i^-(Z_i) \\ &= ms_i(Z_i) \end{aligned}$$

Thus,

$$m_p(Z) \leq \sum_{i=1}^n ms_i(Z_i) + \varepsilon_1.$$

Again, by the regularity of m_p, by the lemma 1.3.3.3 and by 1.3.2.7, there exist $U_1, \dots, U_n \in \mathcal{U}$ pairwise disjoints, such that $Z_i \subset U_i \subset U$ and

$$m_p(U_i) < \frac{\varepsilon_1}{n} + m_p(Z_i).$$

Then,

$$\sum_{i=1}^{n} m_p(U_i \backslash Z_i) < \varepsilon_1.$$

For each $i \in \{1, \dots, n\}$, applying the lemma 1.3.2.6 and taking $h_i \in C_b(X)$ such that $0 \leq h_i \leq \mathscr{X}_X$, $h_i(Z_i) = \{1\}$ and $h_i(X \backslash U_i) = \{0\}$. If we consider the function $f = \sum_{i=1}^{n} h_i \otimes s_i$, then $f \in W$. In fact, if $x \in X$ is such that $f(x) \neq 0$, then there exists $i \in \{1, \dots, n\}$ for which $x \in U_i$. Since the sets U_j are pairwise disjoint, we have that $h_j(x) = 0$, for $j \neq i$. Thereby, $p(f(x)) = p(h_i(x)s_i) \leq p(s_i) \leq 1$.

Then, by hypothesis $\left|\int_X f\, dm\right| \leq \varepsilon$. Since $f = 0$ in $X \backslash (\bigcup_{i=1}^{n} U_i)$ and $f = s_i$ in Z_i, we have that:

$$\begin{aligned}
\left|\int_X f\, dm\right| = \left|\int_{\bigcup_{i=1}^{n} U_i} f\, dm\right| &= \left|\sum_{i=1}^{n} \int_{U_i} f\, dm\right| \\
&= \left|\sum_{i=1}^{n} \int_{Z_i} f\, dm + \sum_{i=1}^{n} \int_{U_i \backslash Z_i} f\, dm\right| \\
&= \left|\sum_{i=1}^{n} ms_i(Z_i) + \sum_{i=1}^{n} \int_{U_i \backslash Z_i} f\, dm\right| \\
&\geq \left|\sum_{i=1}^{n} ms_i(Z_i)\right| - \left|\sum_{i=1}^{n} \int_{U_i \backslash Z_i} f\, dm\right| \\
&\geq m_p(Z) - \varepsilon_1 - \sum_{i=1}^{n} ms_i(U_i \backslash Z_i) \\
&\geq m_p(Z) - 2\varepsilon_1.
\end{aligned}$$

Finally, since $\varepsilon_1 > 0$ is arbitrary, it is concluded that $m_p(Z) \leq \varepsilon$. □

3.3. Integral representation of $\beta_{\mathcal{P}}$-continuous linear functionals on $C_b(X,E)$.

In this section we will prove that each linear $\beta_{\mathcal{P}}$-continuous functional on $C_b(X,E)$, has a representation of integral operator with respect to an only measure in $M_{\mathcal{P},t}(X,E')$. This work is mentioned by J. Zafarani in [31] and it is based on the work by A. Katsaras presented in [15].

The following theorem is one of the first results of integral representation of functionals on $C_b(X)$ and it was presented at the beginning of the 40' by A. Alexandroff.

Theorem 3.3.1 (Alexandroff Representation Theorem)**.**

The linear map $T:(M(X),\|\cdot\|_*)\to(C_b(X),\tau_{\|\cdot\|})'$ *defined by*

$$T(\mu)(f)=\int_X f\,d\mu,$$

is an isomorphism, that is to say, satisfies the equality:

$$\|\mu\|_*=\|T\|_\infty:=\sup\{|T(f)|:\|f\|\leq 1\}.$$

It can be found a self-contained proof of this theorem in [29, page 115, 5.1]. In 1965, V. Varadarajan proved in [28, page 179, theorem 29], the following result:

Theorem 3.3.2. *Let* $\psi\in(C_b(X),\tau_{\|\cdot\|})'$ *and* $\mu\in M(X)$ *be such that* $\psi(f)=\int_X f\,d\mu$, *for each* $f\in C_b(X)$. *Then, the functional* ψ *is tight, if and only if* μ *is a tight measure.*

Then, in 1972, F. Sentilles defined in [24] the topology γ_0 (calling it substrict topology and denoting it by β_0) and proved that $(C_b(X),\gamma_0)'=M_t(X)$ ([24, page 320, theorem 4.4]). Using these three results, we can state the following corollary:

Corollary 3.3.3 (Representation theorem for γ_0-continuous functionals in $C_b(X)$)**.**

The linear map $T:M_t(X)\to(C_b(X),\gamma_0)'$ *defined by*

$$T(\mu)(f)=\int_X f\,d\mu.$$

is bijective and satisfies the equality $\|\mu\|_*=\|T\|_\infty$.

We will denote by $C_{rc}(X,E)$, the function space

$$\{f \in C_b(X,E) : f(X) \text{ is relatively compact}\}.$$

In 1974, A. Katsaras presented in [13], the following generalization of the Alexandroff representation theorem:

Theorem 3.3.4.

The linear map $T : M(X,E') \to (C_{rc}(X,E), \tau_u)'$ *defined by*

$$T(m)(f) = \int_X f\, dm,$$

is bijective.

Two years later, A.Katsaras in [15], defined on the space $C_{rc}(X,E)$ the topology $\beta_{\mathscr{F}}$, which is for us the induced subspace topology by the topology $\beta_{\mathcal{P}}$. We will denote the topology $\beta_{\mathscr{F}}$ simply by $\beta_{\mathcal{P}}$. In said article, it is shown the following representation theorem.

Theorem 3.3.5.

The linear map $T : M_{\mathcal{P},t}(X,E') \to (C_{rc}(X,E), \beta_{\mathcal{P}})'$ *defined by*

$$T(m)(f) = \int_X f\, dm,$$

is bijective.

In the following, we will generalize this result to the dual space $(C_b(X,E), \beta_{\mathcal{P}})'$.

Definition 3.3.6. *Let* $\psi : C_b(X,E) \to \mathbb{R}$ *be a linear functional* τ_u*-continuous and* $p \in s(E)$. *It is define* $\|\psi\|_{p,\infty} := \sup\{|\psi(f)| : \|f\|_p \leq 1\}$.

Proposition 3.3.7. *If* $m \in M_{\mathcal{P},p,t}(X,E'_p)$ *and* $D \in \mathcal{B}$, *then, the map* $\psi_m : C_b(X,E) \to \mathbb{R}$, *defined by*

$$\psi_m(f) := \int_D f\, dm,$$

belongs to the dual $(C_b(X,E), \beta_p(\mathcal{P}))'$. *Also,*

$$\|\psi_m\|_{p,\infty} = m_p(D).$$

Proof. The map ψ_m clearly is linear. We will prove that ψ_m is β_p-continuous. Let $\varepsilon > 0$ and $r > 0$ be such that $r \cdot m_p(X) < \frac{\varepsilon}{2}$. Let $K \in \mathcal{P}$ be such that $m_p(G) < \frac{\varepsilon}{2}$, for each $G \in \mathcal{B}$, $G \subset X \backslash K$. We take $f \in C_b(X,E)$ such that $\|f\|_p \leq 1$ and $\|f\|_{p,K} \leq r$. The set $G := \{x \in X : p(f(x)) \leq r\} \in \mathcal{Z}$, contains K. Therefore, $D\backslash G \subset X\backslash G \subset X\backslash K$.

Applying the lemma 3.2.17, we have that:

$$\left|\int_{D\cap G} f\, dm\right| \leq \sup_{x\in D\cap G}\{p(f(x))\} \cdot m_p(D\cap G) \leq r \cdot m_p(X) \leq \frac{\varepsilon}{2}\,.$$

Analogously,

$$\left|\int_{D\backslash G} f\, dm\right| \leq \|f\|_p \cdot m_p(X\backslash G) \leq \frac{\varepsilon}{2}\,.$$

Thereby,

$$|\psi_m(f)| \leq \left|\int_{D\cap G} f\, dm\right| + \left|\int_{D\backslash G} f\, dm\right| \leq \varepsilon.$$

Then, by lemma 3.1.2, the map ψ_m is $\mathcal{P}_p$-tight. Applying the corollary 3.1.5, it is conclude the β_p-continuity of ψ_m.

Now let us prove the equality $\|\psi_m\|_{p,\infty} = m_p(D)$. If $f \in C_b(X,E)$ is such that $\|f\|_p \leq 1$, then

$$|\psi_m(f)| = \left|\int_D f\, dm\right| \leq \|f\|_p \cdot m_p(D) \leq m_p(D)\,.$$

Therefore, $\|\psi_m\|_{p,\infty} \leq m_p(D)$. To prove that $m_p(D) \leq \|\psi_m\|_{p,\infty}$, due to the definition of the measure m_p and the regularity of the measures $|me|$ for $e \in E$, it is enough to prove that for each $\varepsilon > 0$, $\sum_{i=1}^n me_i(Z_i) \leq \|\psi_m\|_{p,\infty} + \varepsilon$, where $\{Z_i\}_i^n$ is a finite and arbitrary collection of elements of $\mathcal{Z}$, pairwise disjoint such that $Z_i \subset D$ and where $\{e_i\}_i^n$ is a finite and arbitrary collection in E such that $p(e_i) \leq 1$ and $me_i(Z_i) \geq 0$ for $i = 1, \ldots, n$.

Let $\varepsilon > 0$ be and considering collections $\{Z_i\}_i^n$ and $\{e_i\}_i^n$ as the previously mentioned. By the regularity of m_p and by the lemmas 1.3.3.3 and 1.3.2.7, we can choose a collection $\{U_i\}_i^n \subset \mathcal{U}$, pairwise disjoint such that: $Z_i \subset U_i$ and $m_p(U_i) < \frac{\varepsilon}{n} + m_p(Z_i)$ for $i = 1, \ldots, n$.

We define $D_i := D \cap U_i$. Then, $m_p(D_i\backslash Z_i) \leq m_p(U_i\backslash Z_i) < \frac{\varepsilon}{n}$. By the lemma 1.3.2.6, for each $i \in \{1, \ldots, n\}$, we can take $f_i \in C_b(X)$ such that $0 \leq f \leq \mathscr{X}_X$, $f_i(Z_i) = \{1\}$ and $f_i(X\backslash U_i) = \{0\}$.

Since $\sum_{i=1}^n f_i \otimes e_i = 0$ in $X \backslash (\bigcup_{i=1}^n U_i)$ and since the U_i are disjoint, we have that $\| \sum_{i=1}^n f_i \otimes e_i \|_p \leq 1$. Thereby, since $\sum_{i=1}^n f_i \otimes e_i = 0$ in $D \backslash (\bigcup_{i=1}^n D_i)$, we have:

$$\begin{aligned}
\left| \psi_m \left(\sum_{i=1}^n f_i \otimes e_i \right) \right| &= \left| \int_D \sum_{i=1}^n f_i \otimes e_i \, dm \right| \\
&= \left| \int_{\bigcup_{i=1}^n D_i} \sum_{i=1}^n f_i \otimes e_i \, dm \right| \\
&= \left| \sum_{i=1}^n \int_{D_i} f_i \otimes e_i \, dm \right| \\
&= \left| \sum_{i=1}^n \int_{Z_i} f_i \otimes e_i \, dm + \sum_{i=1}^n \int_{D_i \backslash Z_i} f_i \otimes e_i \, dm \right| \\
&\geq \left| \sum_{i=1}^n \int_{Z_i} f_i \otimes e_i \, dm \right| - \left| \sum_{i=1}^n \int_{D_i \backslash Z_i} f_i \otimes e_i \, dm \right| \\
&= \left| \sum_{i=1}^n m e_i(Z_i) \right| - \left| \sum_{i=1}^n \int_{D_i \backslash Z_i} f_i \otimes e_i \, dm \right|
\end{aligned}$$

Then,

$$\begin{aligned}
\sum_{i=1}^n m e_i(Z_i) &\leq \left| \psi_m \left(\sum_{i=1}^n f_i \otimes e_i \right) \right| + \left| \sum_{i=1}^n \int_{D_i \backslash Z_i} f_i \otimes e_i \, dm \right| \\
&\leq \|\psi_m\|_{p,\infty} + \sum_{i=1}^n m_p(D_i \backslash Z_i) \\
&\leq \|\psi_m\|_{p,\infty} + \varepsilon \, .
\end{aligned}$$

□

Proposition 3.3.8. *If $\psi \in (C_b(X, E), \beta_p(\mathcal{P}))'$, then there exists $m \in M_{\mathcal{P},p,t}(X, E'_p)$ such that*

$$\psi(f) = \int_X f \, dm \, .$$

Proof. Let $\psi \in (C_b(X, E), \beta_p)'$. Then, ψ is u_p-continuous, that is to say, $\|\psi\|_{p,\infty} < \infty$.

<u>Step 1:</u> Defining a map $m : \mathcal{B} \to E'_p$.

For each $e \in E$, considering the linear functional $\psi_e : C_b(X) \to \mathbb{R}$, defined by $\psi_e(f) := \psi(f \otimes e)$. If we define the map $S : C_b(X) \to C_b(X, E)$, $S(f) = f \otimes e$, then by the lemma 2.3.6.b, the functional $\psi_e = \psi \circ S$ is $\gamma_{\mathcal{P}}$-continuous.

Therefore $\psi_e \in (C_b(X), \gamma_0)'$. Applying the corollary 3.3.3, there exists $m_e \in M_t(X)$ such that $\psi_e(f) = \int_X f\, dm_e$, for each $f \in C_b(X)$.

Note that for $f \in C_b(X)$, $\|f\| \leq 1$, we have

$$|\psi_e(f)| = |\psi(f \otimes e)| \leq \|\psi\|_{p,\infty} \|f\| p(e) \leq \|\psi\|_{p,\infty} p(e).$$

Then,

$$|m_e|(X) = \|m_e\|_* = \|\psi_e\|_\infty \leq \|\psi\|_{p,\infty} p(e). \tag{3.2}$$

For each $A \in \mathcal{B}$, we define the map $m(A) : E \to \mathbb{R}$ by $\langle m(A), e\rangle := m_e(A)$. We will prove that $m(A) \in E'_p$. Let $e_1, e_2 \in E$ and $\lambda \in \mathbb{R}$. For each $f \in C_b(X)$, we have that $\psi_{e_1+\lambda e_2}(f) = \psi(f \otimes (e_1 + \lambda e_2)) = \psi(f \otimes e_1) + \lambda\psi(f \otimes e_2) = \psi_{e_1}(f) + \lambda\psi_{e_2}(f)$.

Thereby, for each $f \in C_b(X)$,

$$\int_X f\, dm_{e_1+\lambda e_2} = \int_X f\, dm_{e_1} + \lambda \int_X f\, dm_{e_2}\,.$$

We will use the following abbreviations: $m_1 = m_{e_1}$, $m_2 = m_{e_2}$ and $m_3 = m_{e_1+\lambda e_2}$. Let $Z \in \mathcal{Z}$ and $\varepsilon > 0$. By the regularity of the measures $|m_1|, |m_2|$ and $|m_3|$, there exist U_1, U_2 and U_3 in $\mathcal{U}$ such that $Z \subset U_i$ and $|m_i|(U_i \backslash Z) < \varepsilon$ $(i = 1, 2, 3)$. Consider $U := U_1 \cap U_2 \cap U_3$ and $g \in C_b(X)$ such that $0 \leq g \leq \mathscr{X}_X$, $Z = g^{-1}(\{1\})$ and $X \backslash U = g^{-1}(\{0\})$.

We have that,

$$\begin{aligned}
&\int_X g\, dm_3 = \int_X g\, dm_1 + \lambda \int_X g\, dm_2 \\
\Longrightarrow &\int_U g\, dm_3 = \int_U g\, dm_1 + \lambda \int_U g\, dm_2 \\
\Longrightarrow &\int_Z g\, dm_3 + \int_{U\backslash Z} g\, dm_3 = \int_Z g\, dm_1 + \int_{U\backslash Z} g\, dm_1 + \lambda \int_Z g\, dm_2 + \lambda \int_{U\backslash Z} g\, dm_2 \\
\Longrightarrow &\, m_3(Z) - m_1(Z) - \lambda m_2(Z) = \int_{U\backslash Z} g\, dm_1 + \lambda \int_{U\backslash Z} g\, dm_2 - \int_{U\backslash Z} g\, dm_3
\end{aligned}$$

$$\Longrightarrow |m_3(Z) - m_1(Z) - \lambda m_2(Z)| \leq |m_1|(U\backslash Z) + |\lambda||m_2|(U\backslash Z) + |m_3|(U\backslash Z)$$

$$\Longrightarrow |m_3(Z) - m_1(Z) - \lambda m_2(Z)| \leq \varepsilon + |\lambda|\varepsilon + \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we have that $m_3(Z) = m_1(Z) + \lambda m_2(Z)$, for each $Z \in \mathcal{Z}$. Therefore, $m_3 = m_1 + \lambda m_2$, that is to say, $m_{e_1+\lambda e_2} = m_{e_1} + \lambda m_{e_2}$.

Then, $\langle m(A), e_1 + \lambda e_2\rangle = \langle m(A), e_1\rangle + \lambda\langle m(A), e_2\rangle$, proving the linearity of $m(A)$. We affirm that $m(A)$ is p-continuous. In fact, by (3.2), we have that $|\langle m(A), e\rangle| = |m_e(A)| \leq |m_e|(A) \leq |m_e|(X) \leq \|\psi\|_{p,\infty} p(e)$.

We define the map $m : \mathcal{B} \to E'_p$, $A \mapsto m(A)$.

<u>STEP 2:</u> We will prove that $m \in M_p(X, E'_p)$.

By construction, for each $e \in E$, we have $me = m_e \in M_t(X) \subset M(X)$.

First we will prove that $m_p(X) \leq \|\psi\|_{p,\infty}$. Due to the definition of the measure m_p and by the regularity of the measures $|me|$ for $e \in E$, is enough to prove that for each $\varepsilon > 0$, $\sum_{i=1}^n me_i(Z_i) \leq \|\psi\|_{p,\infty} + \varepsilon$, where $\{Z_i\}_i^n$ is a finite and arbitrary collection of elements of $\mathcal{Z}$ pairwise disjoint and $\{e_i\}_i^n$ is a finite and arbitrary collection in E such that $p(e_i) \leq 1$ and $me_i(Z_i) \geq 0$ for $i = 1, \ldots, n$.

Let $\{Z_i\}_{i=1}^n$ and $\{e_i\}_{i=1}^n$ be collections as mentioned above. Let $\varepsilon > 0$. By the regularity of m_p and by the lemmas 1.3.3.3 and 1.3.2.7, we can choose a collection $\{U_i\}_i^n \subset \mathcal{U}$, pairwise disjoint such that: $Z_i \subset D_i$ and $m_p(U_i) < \frac{\varepsilon}{n} + m_p(Z_i)$ for $i = 1, \ldots, n$.

Then, $m_p(U_i \backslash Z_i) < \frac{\varepsilon}{n}$. By the lemma 1.3.2.6, for each $i \in \{1, \ldots, n\}$, we can take $f_i \in C_b(X)$ such that $0 \leq f \leq \mathscr{X}_X$, $f_i(Z_i) = \{1\}$ and $f_i(X \backslash U_i) = \{0\}$.

Since $\sum_{i=1}^n f_i = 0$ in $X \backslash (\bigcup_{i=1}^n U_i)$ and since U_i are disjoint, we have that $\| \sum_{i=1}^n f_i \otimes e_i\|_p \leq 1$. Thereby,

$$
\begin{aligned}
\left|\sum_{i=1}^n \psi_{e_i}(f_i)\right| &= \left|\sum_{i=1}^n \int_X f_i\, dm_{e_i}\right| \\
&= \left|\sum_{i=1}^n \int_{U_i} f_i\, dm_{e_i}\right| \\
&= \left|\sum_{i=1}^n \int_{Z_i} f_i\, dm_{e_i} + \sum_{i=1}^n \int_{U_i \backslash Z_i} f_i\, dm_{e_i}\right| \\
&\geq \left|\sum_{i=1}^n \int_{Z_i} f_i\, dm_{e_i}\right| - \left|\sum_{i=1}^n \int_{U_i \backslash Z_i} f_i\, dm_{e_i}\right| \\
&= \left|\sum_{i=1}^n me_i(Z_i)\right| - \left|\sum_{i=1}^n \int_{U_i \backslash Z_i} f_i\, dme_i\right|
\end{aligned}
$$

Then,

$$\begin{aligned}\sum_{i=1}^{n} me_i(Z_i) &\leq \left|\sum_{i=1}^{n} \psi_{e_i}(f_i)\right| + \left|\sum_{i=1}^{n} \int_{U_i \setminus Z_i} f_i \, dme_i\right| \\ &\leq \left|\sum_{i=1}^{n} \psi(f_i \otimes e_i)\right| + \sum_{i=1}^{n} |me_i|(U_i \setminus Z_i) \\ &\leq \|\psi\|_{p,\infty} + \varepsilon\end{aligned}$$

By the observation 3.2.5, we have that m is finitely additive. Then, it is concluded that $m \in M_p(X, E'_p)$.

<u>STEP 3:</u> We will show that $\psi(f) = \int_X f \, dm$, for each $f \in C_b(X, E)$.

Let $f \in C_b(X)$, $e \in E$ and $\alpha = \{A_1, \ldots, A_n, s_1, \ldots, s_n\} \in \Omega_X$. It is satisfy the equality

$$\begin{aligned}\omega_\alpha(f, m_e) = \sum_{i=1}^{n} m_e(A_i) f(s_i) &= \sum_{i=1}^{n} \langle m(A_i), e\rangle f(s_i) \\ &= \sum_{i=1}^{n} \langle m(A_i), f(s_i) e\rangle \\ &= \sum_{i=1}^{n} \langle m(A_i), (f \otimes e)(s_i)\rangle = \omega_\alpha(f \otimes e, m).\end{aligned}$$

Therefore,

$$\psi(f \otimes e) = \psi_e(f) = \int_X f \, dm_e = \int_X f \otimes e \, dm \, .$$

Then, we can conclude

$$\psi(g) = \int_X g \, dm \, ,$$

for each $g \in C_b(X) \otimes E$.

Since $me \in M_t(X)$, for each $e \in E$, by the result 3.2.12, we have that $m_p \in M_t(X)$ and by 3.3.7, the functional $\psi_m : C_b(X, E) \to \mathbb{R}$, $\psi_m(f) = \int_X f \, dm$ is $\beta_p(K(X))$-continuous. Due to the space $C_b(X) \otimes E$ is $\beta_p(K(X))$-dense in $C_b(X, E)$ and $\psi(g) = \psi_m(g)$ for each $g \in C_b(X) \otimes E$, it follows that $\psi(f) = \int_X f \, dm$, for all $f \in C_b(X, E)$.

<u>STEP 4</u>: We will prove that $m_p \in M_{\mathcal{P},t}(X)$.

Let $\varepsilon > 0$ and $V := \{f \in C_b(X,E) : |\int_X f\,dm| \leq \varepsilon\}$. By the $\beta_p(\mathcal{P})$-continuity of ψ, it follows that V is a $\beta_p(\mathcal{P})$-neighborhood of 0. Since the topologies $\beta_p(\mathcal{P})$ and $\tau_p(\mathcal{P})$ coincide in the u_p-bounded sets, there exist $K \in \mathcal{P}$ and $\delta > 0$ such that:

$$\{f \in C_b(X,E) : \|f\|_{p,K} \leq \delta, \|f\|_p \leq 1\} \subset V.$$

Since the set W of the lemma 3.2.19 is contained in V, we have that $m_p(G) \leq \varepsilon$ for each $G \in \mathcal{B}$, $G \subset X \backslash K$. Therefore $m_p \in M_{\mathcal{P},t}(X)$. □

As a consequence of the two previously results, we have the following corollary.

Corollary 3.3.9. *The linear map* $T : M_{\mathcal{P},p,t}(X, E'_p) \to (C_b(X,E), \beta_p(\mathcal{P}))'$, *defined by*

$$T(m)(f) = \int_X f\,dm$$

is bijective and satisfies the equality

$$m_p(X) = \|T(m)\|_{p,\infty}.$$

Proof. By 3.3.7, the map T is well defined and satisfies the equality $m_p(X) = \|T(m)\|_{p,\infty}$ for each $m \in M_{\mathcal{P},p,t}(X, E'_p)$. The surjection of T is due to 3.3.8. The linearity of T is clear.

We will show that T is injective. Let $m \in M_{\mathcal{P},p,t}(X, E'_p)$, such that $T(m) = 0$, that is to say, $\int_X f\,dm = 0$, for each $f \in C_b(X,E)$. Then, $0 = \|T(m)\|_{p,\infty} = m_p(X)$. Let $G \in \mathcal{B}$ and $s \in E$. If $p(s) = 0$, then, $|\langle m(G), s\rangle| \leq m_p(G) \leq m_p(X) = 0$. If $p(s) \neq 0$, then $|\langle m(G), \frac{s}{p(s)}\rangle| \leq m_p(G) \leq m_p(X) = 0$.

Therefore, for each $s \in E$, we have that $\langle m(G), s\rangle = 0$. Thereby, $m(G) = 0$. The result follows from the arbitrariness of $G \in \mathcal{B}$. □

Finally, we can characterize the dual of $(C_b(X,E), \beta_{\mathcal{P}})$ as a measure space.

Corollary 3.3.10. $(C_b(X,E), \beta_{\mathcal{P}})' = \bigcup_{p \in \mathcal{S}(E)} M_{\mathcal{P},p,t}(X, E'_p) = M_{\mathcal{P},t}(X, E')$.

Proof. It is immediate consequence of the corollaries 3.1.7 and 3.3.9. □

3.4. Equicontinuity in space $(C_b(X,E),\beta_{\mathcal{P}})'$.

In this section we will characterize the $\beta_{\mathcal{P}}$-equicontinuos subsets of $(C_b(X,E),\beta_{\mathcal{P}})'$.

Definition 3.4.1. *A subset H of $M(X)$ is called $\mathcal{P}$-tight if it satisfies the following conditions:*

(a) H is bounded by the norm in $M(X)$.

(b) Given $\varepsilon > 0$, there exists $K \in \mathcal{P}$ such that $|\mu|(G) < \varepsilon$ for each $\mu \in H$ and each $G \in \mathcal{B}$, $G \subset X\backslash K$.

This definition relates to the definition of measurement $\mathcal{P}$-tight in the following form: for $\mu \in M(X)$, μ is $\mathcal{P}$-tight if and only if $\{\mu\}$ is $\mathcal{P}$-tight.

Definition 3.4.2. *A subset H of $M(X,E')$ is called $\mathcal{P}$-tight if there exists $p \in s(E)$ such that:*

(a) $H \subset M_p(X,E'_p)$ and

(b) The set $H_p := \{m_p : m \in H\}$ is $\mathcal{P}$-tight in $M(X)$.

Note that for $m \in M(X,E')$, the set $\{m\}$ is $\mathcal{P}$-tight if and only if there exists $p \in s(E)$ such that $m \in M_{\mathcal{P},p,t}(X,E'_p)$. In other words, the space $M_{\mathcal{P},t}(X,E')$ is the set of all the measures $m \in M(X,E')$, such that $\{m\}$ is $\mathcal{P}$-tight.

Theorem 3.4.3. *For all subset H of $M_{\mathcal{P},t}(X,E') = (C_b(X,E),\beta_{\mathcal{P}})'$, the following statements are equivalent:*

(a) H is $\beta_{\mathcal{P}}$-equicontinuous.

(b) H is $\mathcal{P}$-tight.

(c) There exists $p \in s(E)$ and $r > 0$ such that:

(i) $H \subset M_p(X,E'_p)$,

(ii) $\|m_p\|_* \leq r$, for each $m \in H$ and

(iii) Given $\varepsilon > 0$, there exists $K \in \mathcal{P}$ such that $|\int_X f, dm| \leq \varepsilon$, for each $m \in H$ and each $f \in C_b(X,E)$, with $\|f\|_p \leq 1$ and $f = 0$ in K.

Proof. Due to 3.2.19, the statements (b) and (c) are equivalents.

$(a) \Rightarrow (b)$: Suppose that H is $\beta_{\mathcal{P}}$-equicontinuous. Then, there exists $p \in s(E)$ such that H is β_p-equicontinuous and thereby it follows that $H \subset M_p(X, E'_p)$. Let $H_0 = \{f \in C_b(X,E) : |\psi(f)| \leq 1, \forall \psi \in H\}$ the polar of H, respect to the duality $\langle (C_b(X,E),\beta_{\mathcal{P}}), M_{\mathcal{P},t}(X,E')\rangle$.

Then, H_0 is a β_p-neighborhood of 0. Let $\varepsilon > 0$. By 2.7.5, there exist $K \in \mathcal{P}$ and $\delta > 0$, such that:

$$V := \{f \in C_b(X,E) : \|f\|_p \leq 1, \|f\|_{p,K} \leq \delta\} \subset \varepsilon H_0.$$

From this, it follows that $|\int_X f\, dm| \leq \varepsilon$, for each $m \in H$, and each $f \in C_b(X,E)$ such that $\|f\|_p \leq 1$ and $f = 0$ in K. Applying 3.2.19, we have that $m_p(G) \leq \varepsilon$, for $G \in \mathcal{B}$, $G \subset X \backslash K$ and each $m \in H$. It now remains to prove that $\sup\{\|m_p\|_* : m \in H\} < \infty$.

Let $K_0 \in \mathcal{P}$ and $0 < \delta_0$ such that:

$$W := \{f \in C_b(X,E) : \|f\|_p \leq 1, \|f\|_{p,K_0} \leq \delta_0\} \subset H_0.$$

Note that if $\|f\|_p \leq \delta_0$, then $f \in W$ and from this we have that $|\int_X f\, dm| \leq 1$ for each $m \in H$. Thereby, for each $m \in H$ we have

$$\|m_p\|_* = m_p(X) = \sup\left\{\left|\int_X f\, dm\right| : \|f\|_p \leq 1\right\} \leq \frac{1}{\delta_0}.$$

$(b) \Rightarrow (a)$: Let $p \in s(E)$ be such that $H \subset M_p(X, E'_p)$ and $\{m_p : m \in H\}$ is $\mathcal{P}$-tight. Let $d = \sup\{\|m_p\|_* : m \in H\}$. Given $r > 0$, there exists $K \in \mathcal{P}$, such that: $m_p(G) \leq \frac{1}{2r}$, for each $m \in H$ and each $G \in \mathcal{B}$, $G \subset X \backslash K$.

Let $V := \{f \in C_b(X,E) : \|f\|_p \leq r, \|f\|_{p,K} \leq \frac{1}{2d}\}$. If $f \in V$ and $m \in H$, then, $|\int_X f\, dm| \leq 1$. In fact, let $N := \{x \in X : p(f(x)) > \frac{1}{2d}\}$. We have that $N \in \mathcal{U}$ and $N \subset X \backslash K$. Then, $m_p(N) \leq \frac{1}{2r}$. Thereby,

$$\left|\int_X f\, dm\right| \leq \left|\int_N f\, dm\right| + \left|\int_{X\backslash N} f\, dm\right| \leq \|f\|_p m_p(N) + \frac{1}{2d} m_p(X \backslash N) \leq r \cdot \frac{1}{2r} + \frac{1}{2d} \cdot d = 1.$$

This proves that $V \subset H_0$ and then by 2.7.5, H_0 is a β_p-neighborhood of 0. Therefore, H_0 is a $\beta_{\mathcal{P}}$-neighborhood of 0, completing the proof. □

3.5. Linear Operators in $(C_b(X), \gamma_{\mathcal{P}})$.

Let F be a Frechet space which topology will be denote by τ_F. Let τ be a vector topology in $C_b(X)$.

Definition 3.5.1. *An operator $T : C_b(X) \to F$ is (τ, τ_F)-weakly compact if there exists a τ-neighborhood V of zero such that $T(V)$ is relatively $\sigma(F, F')$-compact. We will simply say that T is weakly compact if it is $(\tau_{\|\cdot\|}, \tau_F)$-weakly compact.*

Proposition 3.5.2. *For a linear operator $T : C_b(X) \to E$ the following statements are equivalents:*

(a) T is weakly compact and $(\gamma_{\mathcal{P}}, \tau_F)$-continuous.

(b) T is $(\gamma_{\mathcal{P}}, \tau_F)$-weakly compact.

Proof. $(a) \Rightarrow (b)$: There exists a $\tau_{\|\cdot\|}$-neighborhood V of 0 in $C_b(x)$ such that $T(V)$ is relatively $\sigma(F, F')$-compact. If $B \subset C_b(X)$ is $\tau_{\|\cdot\|}$-bounded, then there exists $n \in \mathbb{N}$ such that $T(B) \subset T(nV)$. Then, $T(B)$ is relatively $\sigma(F, F')$-compact. Thereby, T carries sets $\gamma_{\mathcal{P}}$-bounded in sets relatively $\sigma(F, F')$-compacts. Applying 2.5.8, we obtain that $(C_b(X), \gamma_{\mathcal{P}})$ is a gDF-space, since $\mathbb{R}$ is a gDF-space. Then, by [19, page 435, Theorem 3.1] it is concluded that T is an operator $(\gamma_{\mathcal{P}}, \tau_F)$-weakly compact.

$(b) \Rightarrow (a)$: If T is $(\gamma_{\mathcal{P}}, \tau_F)$-weakly compact, then there exists a $\gamma_{\mathcal{P}}$-neighborhood V of zero, such that $T(V)$ is relatively $\sigma(F, F')$-compact. Since $\gamma_{\mathcal{P}} \leq \tau_{\|\cdot\|}$, then T is weakly compact. Is clear that T is $(\gamma_{\mathcal{P}}, \tau_F)$-continuous. □

Analogously and applying the result [19, page 436, note (a)], we have:

Proposition 3.5.3. *For a linear operator $T : C_b(X) \to E$ the following statements are equivalents:*

(a) T is compact and $(\gamma_{\mathcal{P}}, \tau_F)$-continuous.

(b) T is $(\gamma_{\mathcal{P}}, \tau_F)$-compact.

3.6. Integral representation of $(\beta_{\mathcal{P}}, \tau_F)$-weakly compact operators.

In this section we will present a more general definition of weakly compact operator than the one given in section 3.5 and we will prove that each linear, $(\beta_{\mathcal{P}}, \tau_F)$-weakly compact and $(\beta_{\mathcal{P}}, \tau_F)$-continuous operator $T : C_b(X, E) \to F$, has an only representation of integral operator with respect to an only measure in certain space of measures.

Let F be a Hausdorff locally convex. Consider the space $\mathscr{L}(E, F)$ of all the linear and (τ_E, τ_F)-continuous operators of E in F. For each $q \in s(F)$, we will denote by F'_q to the dual space $(F, \{q\})'$. Then, for each $x' \in F'$, there exists $q \in s(F)$, such that $x' \in F'_q$. For each $q \in s(F)$ and each $x' \in F'_q$, it is define the seminorm $\|x'\|^{q,\infty} := \sup\{|\langle x', s\rangle| : q(s) \leq 1\}$. The family of all the bounded subsets of E will be denoted by $Bd(E)$.

Definition 3.6.1. *It is define the space $M_{\mathcal{P},t}(X, \mathscr{L}(E, F))$ as the family of all the finitely additive set functions $m : \mathcal{B} \to \mathscr{L}(E, F)$ that satisfy the following conditions:*

(I) for each $x' \in F'$, the set function $x'm : \mathcal{B} \to E'$, defined by $\langle (x'm)(G), s\rangle := \langle x', m(G)(s)\rangle$, belongs to $M_{\mathcal{P},t}(X, E')$.

(II) Given $q \in s(F)$, there exists $p \in s(E)$ such that:

i. For each $x' \in F'$, with $\|x'\|^{q,\infty} \leq 1$, we have that $x'm \in M_{\mathcal{P},p,t}(X, E'_p)$ and

ii. $m^q_p(X) < \infty$, where for each $G \in \mathcal{B}$ it is define:

$$m^q_p(G) := \sup\{(x'm)_p(G) : \|x'\|^{q,\infty} \leq 1\}.$$

Definition 3.6.2. *It is define the space $\mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E, F))$ as the family of all the set functions $m \in M_{\mathcal{P},t}(X, \mathscr{L}(E, F))$ that satisfy the condition (R), defined by:*

(R): For each $S \in Bd(E)$, the set

$$V_{m,S} := \left\{\sum_{i=1}^{n} m(G_i)(s_i) : n \in \mathbb{N}, \{G_i\}_{i=1}^{n} \in \Pi(X), s_i \in S\right\}$$

is relatively weakly compact in F.

Definition 3.6.3. *Let $m \in M_{\mathcal{P},t}(X, \mathscr{L}(E,F))$, $G \in \mathcal{B}$ and $f : X \to E$ be an arbitrary function. We will say that f is* ***m-integrable in G*** *if satisfies the following conditions:*

(a) For each $x' \in F'$, there exists the integral $\int_G f\, d(x'm)$,

(b) There exists a vector in F denoted by $\int_G f\, dm$ such that for each $x' \in F'$, we have

$$\left\langle x', \int_G f\, dm \right\rangle = \int_G f\, d(x'm).$$

We will say that ***f is m-integrable****, if f is m-integrable in each $G \in \mathcal{B}$.*

Remark 3.6.4. *Note that for each function of the form $e\mathscr{X}_G$, with $e \in E$, $G \in \mathcal{B}$ is m-integrable. On the other hand, since F is a Hausdorff space, by the Hahn-Banach Theorem, it is concluded that: if the vector $\int_G f\, dm$ exists, then is unique.*

Lemma 3.6.5. *Given $G \in \mathcal{B}$, $m \in M_{\mathcal{P},t}(X, \mathscr{L}(E,F))$ and $f \in C_b(X,E)$ we have the existence of $\int_G f d\,(x'm)$ for each $x' \in F'$.*

Proof. Let $x' \in F'$. Then, there exists $p \in s(E)$ such that $x'm \in M_{\mathcal{P},p,t}(X, E'_p)$.

Applying 3.2.17, we obtain that f is $(x'm)$-integrable in G, that s to say, the integral $\int_G f d\,(x'm)$ exists. ☐

Lemma 3.6.6. *Let $G \in \mathcal{B}$, $m \in M_{\mathcal{P},t}(X, \mathscr{L}(E,F))$ and $f \in C_b(X,E)$. The function f is m-integrable in G if and only if there exists $z \in F$ such that $x'(z) = \int_G f d(x'm)$, $\forall x' \in F'$.*

Proof. By the previous lemma. ☐

Definition 3.6.7. *Let A and B be topological vector spaces and let $Bd(A)$ be the family of all the τ_A-bounded subsets of A. We will say that a linear operator $T : A \to B$ is* ***(τ_A, τ_B)-weakly compact****, if $T(S)$ is relatively $\sigma(B, B')$-compact, for each $S \in Bd(A)$.*

Lemma 3.6.8. *Let $T : A \to B$ be a linear operator between topological vector spaces. If $i : B \to B''$ is the canonic injection, then, are equivalents:*

(i) T *is* (τ_A, τ_B)*-weakly compact*

(ii) $T''(A'') \subset i(B)$

(iii) $T' : B' \to A'$ *is* $(\tau(B', B), \beta(A', A))$*-continuous.*

Proof. By [11, page 131, lemma 1]. □

Lemma 3.6.9. *Let* $f_0 \in C_b(X, E)$ *and* $G \in \mathcal{B}$*. The map* $\phi : M_{\mathcal{P},t}(X, E') \to \mathbb{R}$*, defined by*

$$\phi(m) = \int_G f_0 \, dm$$

is $\beta((C_b(X, E), \beta_{\mathcal{P}})', (C_b(X, E), \beta_{\mathcal{P}}))$*-continuous. That is to say,* $\phi \in (C_b(X, E), \beta_{\mathcal{P}})''$*.*

Proof. Let $A = \{f \in C_b(X, E) : ||f||_p \leq ||f_0||_p, \forall p \in s(E)\}$. Then A is a τ_u-bounded set and, therefore, we have that A is $\beta_{\mathcal{P}}$-bounded. Therefore, the polar A^0 in $(C_b(X, E), \beta_{\mathcal{P}})'$ is a $\beta((C_b(X, E), \beta_{\mathcal{P}})', (C_b(X, E), \beta_{\mathcal{P}}))$-neighborhood of zero.

We will prove that ϕ is bounded in A^0. Let $m \in A^0$ and $\varepsilon > 0$. There exists $\{G_1, \ldots, G_n, x_1, \ldots, x_n\} \in \Omega_X$ such that

$$\left| \int_X f_0 \, dm \right| \leq \left| \sum_{i=1}^n \langle m(G_i), s_i \rangle \right| + \varepsilon,$$

where $s_i = f(x_i)$. By the regularity of each ms_i, we can choose $\{Z_1, \ldots, Z_n\} \subset \mathcal{Z}$ such that $Z_i \subset G_i$ and

$$\left| \left| \sum_{i=1}^n \langle m(G_i), s_i \rangle \right| - \left| \sum_{i=1}^n \langle m(Z_i), s_i \rangle \right| \right| \leq \left| \sum_{i=1}^n ms_i(G_i \backslash Z_i) \right| \leq \sum_{i=1}^n |ms_i(G_i \backslash Z_i)| \leq \varepsilon.$$

Then,

$$\left| \sum_{i=1}^n \langle m(G_i), s_i \rangle \right| \leq \left| \sum_{i=1}^n ms_i(Z_i) \right| + \varepsilon$$

Now, by the regularity of each $|ms_i|$, we can choose $\{U_1, \ldots, U_n\} \subset \mathcal{U}$ such that $Z_i \subset U_i$, $U_i \cap U_j = \emptyset$ if $i \neq j$ and $\sum_{i=1}^n |ms_i|(U_i - Z_i) < \varepsilon$.

For each $i \in \{1, \ldots, n\}$, we choose $h_i \in C_b(X)$ with $0 \leq h_i \leq \mathscr{X}_X$, $h_i^{-1}\{1\} = Z_i$ and $h_i^{-1}\{0\} = X \backslash U_i$. If $h = \sum_{i=1}^n h_i \otimes s_i$, then, we have that $h \in A$, therefore $|\int_X h \, dm| \leq 1$.

Therefore we conclude that

$$\left|\sum_{i=1}^{n} m(Z_i)(s_i)\right| - \varepsilon \geq \left|\int_X f_0 \, dm\right| - 3\varepsilon.$$

On the other hand,

$$\begin{aligned}
1 \geq \left|\int_X h \, dm\right| &\geq \left|\sum_{i=1}^{n} \int_{Z_i} h_i \otimes s_i \, dm\right| - \left|\sum_{i=1}^{n} \int_{U_i - Z_i} h_i \otimes s_i \, dm\right| \\
&\geq \left|\sum_{i=1}^{n} m(Z_i)(s_i)\right| - \sum_{i=1}^{n} \left|\int_{U_i - Z_i} h_i \otimes s_i \, dm\right| \\
&= \left|\sum_{i=1}^{n} m(Z_i)(s_i)\right| - \sum_{i=1}^{n} \left|\int_{U_i - Z_i} h_i \, d(ms_i)\right| \\
&\geq \left|\sum_{i=1}^{n} m(Z_i)(s_i)\right| - \sum_{i=1}^{n} |ms_i|(U_i - Z_i) \\
&\geq \left|\sum_{i=1}^{n} m(Z_i)(s_i)\right| - \varepsilon
\end{aligned}$$

Then, $|\int_X f_0 \, dm| - 3\varepsilon \leq 1$. Thereby, by the arbitrariness of $\varepsilon > 0$, it is concluded that

$$|\phi(m)| = \left|\int_X f_0 \, dm\right| \leq 1,$$

proving the continuity of ϕ. □

In the following, the map $i : F \to F''$ will be the canonic injection and $j : i(F) \to F$ will be the map such that $j \circ i = Id_F$ and the map $i_0 : (C_b(X, E), \beta_{\mathcal{P}}) \to (C_b(X, E), \beta_{\mathcal{P}})''$ will be the canonic injection.

Theorem 3.6.10. *If $T : C_b(X, E) \to F$ is an operator $(\beta_{\mathcal{P}}, \tau_F)$-continuous and $(\beta_{\mathcal{P}}, \tau_F)$-weakly compact, then, there exists a unique $m \in \mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E, F))$ such that each $f \in C_b(X, E)$ is m-integrable and $T(f) = \int_X f \, dm$.*

Also, this measure m satisfies the following conditions:

(a) If $p \in s(E)$ and $q \in s(F)$ are such that $||T||_p^q := sup\{q(T(f)) : ||f||_p \leq 1\} < \infty$, then, $m_p^q(X) = ||T||_p^q$.

(b) $\forall x' \in F'$, $T'x' = x'm$.

Conversely, if $m \in \mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E,F))$*, then each* $f \in C_b(X,E)$ *is* m*-integrable and the operator* $T : C_b(X,E) \to F$*, defined by* $T(f) = \int_X f\, dm$*, is* $(\beta_{\mathcal{P}}, \tau_F)$*-continuous and* $(\beta_{\mathcal{P}}, \tau_F)$*-weakly compact.*

Proof. When mention is made of conditions (I), (II) and (R), we are referring to the terms of the definitions 3.6.1 and 3.6.2. Suppose that T is an operator $(\beta_{\mathcal{P}}, \tau_F)$-continuous and $(\beta_{\mathcal{P}}, \tau_F)$-weakly compact. By the lemma 3.6.8, we have that $T''((C_b(X,E), \beta_{\mathcal{P}})'') \subset i(F)$.

By the lemma 3.6.9, if $f \in C_b(X,E)$ and $G \in \mathcal{B}$, then, the function $f\mathscr{X}_G$ define an element of $(C_b(X,E), \beta_{\mathcal{P}})''$ as follows:

$$\langle m, f\mathscr{X}_G \rangle := \int_G f\, dm\,,$$

for each $m \in M_{\mathcal{P},t}(X, E') = (C_b(X,E), \beta_{\mathcal{P}})'$.

<u>Step 1:</u> We will build a set function $m : \mathcal{B} \to \mathscr{L}(E,F)$.

We define $m(G) : E \to F$ by $m(G)(s) := j(T''(s\mathscr{X}_G))$. We have that $m(G) \in \mathscr{L}(E,F)$. In fact, the linearity of $m(G)$ is immediate. By hypothesis and by 3.1.3, given $q \in s(F)$, there exists $p \in s(E)$ such that the operator T is (β_p, q)-continuous. Then $\|T\|_p^q < \infty$.

- For $x' \in (C_b(X,E), u_p)'$, we define $\|x'\|_{p,\infty} := \sup\{|\langle x', f\rangle| : \|f\|_p \le 1\}$.
- For $x'' \in ((C_b(X,E), u_p)', \|\cdot\|_{p,\infty})' =: (C_b(X,E), u_p)''$, we define $\|x''\|_{p,*} := \sup\{|\langle x'', x'\rangle| : x' \in (C_b(X,E), u_p)', \|x'\|_{p,\infty} \le 1\}$.
- For $y' \in F_p'$, we define $\|y'\|^{q,\infty} := \sup\{|\langle y', s\rangle| : q(s) \le 1\}$.
- For $y'' \in (F_q', \|\cdot\|^{q,\infty})' =: F_q''$, we define $\|y''\|^{q,*} := \sup\{|\langle y'', y'\rangle| : y' \in F', \|y'\|^{q,\infty} \le 1\}$.

Applying the Hahn-Banach Theorem, we obtain:

- $\|x'\|_{p,\infty} = \sup\{|\langle x'', x'\rangle| : x'' \in (C_b(X,E), u_p)'', \|x''\|_{p,*} \le 1\}$.
- $q(s) = \sup\{|\langle y', s\rangle| : y' \in F_q', \|y'\|^{q,\infty} \le 1\} = \sup\{|\langle y', s\rangle| : y' \in F', \|y'\|^{q,\infty} \le 1\}$.

- $\|y'\|^{q,\infty} = \sup\{|\langle y'', y'\rangle| : y'' \in F''_q, \|y''\|^{q,*} \leq 1\}$.

Thereby, we have the following equalities:

$$
\begin{aligned}
\|T\|_p^q &= \sup\{q(T(f)) : f \in C_b(X,E), \|f\|_p \leq 1\} \\
&= \sup\{|\langle y', T(f)\rangle| : f \in C_b(X,E), \|f\|_p \leq 1, y' \in F'_q, \|y'\|^{q,\infty} \leq 1\} \\
&= \sup\{|\langle T'(y'), f\rangle| : f \in C_b(X,E), \|f\|_p \leq 1, y' \in F'_q, \|y'\|^{q,\infty} \leq 1\} \\
&= \sup\{\|\langle T'(y')\|_{p,\infty} : y' \in F'_q, \|y'\|^{q,\infty} \leq 1\} \\
&= \sup\{|\langle x'', T'(y')\rangle| : y' \in F'_q, \|y'\|^q \leq 1, x'' \in (C_b(X,E), u_p)'', \|x''\|_{p,*} \leq 1\} \\
&= \sup\{|\langle T''(x''), y'\rangle| : y' \in F'_q, \|y'\|^q \leq 1, x'' \in (C_b(X,E), u_p)'', \|x''\|_{p,*} \leq 1\} \\
&= \sup\{\|T''(x'')\|^{q,*} : x'' \in (C_b(X,E), u_p)'', \|x''\|_{p,*} \leq 1\} \\
&=: \|T''\|_p^q.
\end{aligned}
$$

Note that $||j||^q := sup\{q(j(y'')) : y'' \in i(F), ||y''||^q \leq 1\} = 1$. Then,

$$
\begin{aligned}
q(m(G)(s)) &= q(j(T''(s\mathscr{X}_G))) \\
&\leq ||j||^q \cdot ||T''(s\mathscr{X}_G)||^{q,\infty} \\
&= ||T''(s\mathscr{X}_G)||^{q,\infty} \\
&\leq ||T''||_p^q \cdot ||s\mathscr{X}_G||_{p,*} \\
&\leq ||T''||_p^q \cdot p(s).
\end{aligned}
$$

Therefore $m(G) \in \mathscr{L}(E,F)$. Thus we can define $m : \mathcal{B} \to \mathscr{L}(E,F)$ which is finitely additive.

<u>Step 2:</u> We will prove that $m \in M_{\mathcal{P},t}(X, \mathscr{L}(E,F))$.

If $x' \in F'$ and $s \in E$, then

$$
\begin{aligned}
\langle (x'm)(G), s\rangle &= \langle x', m(G)(s)\rangle \\
&= \langle x', j(T''(s\mathscr{X}_G))\rangle \\
&= \langle T''(s\mathscr{X}_G), x'\rangle \\
&= \langle s\mathscr{X}_G, T'(x')\rangle \\
&= \int_G s\, d(T'(x')) = \langle T'x'(G), s\rangle
\end{aligned}
$$

By the arbitrariness of $s \in E$ and $G \in \mathcal{B}$ it is concluded that $x'm = T'x' \in M_{\mathcal{P},t}(X, E')$. We have proved (b) and the condition (I).

Let $x' \in F'$ be such that $\|x'\|^{q,\infty} \leq 1$. For $f \in C_b(X, E)$ with $\|f\|_p \leq 1$, we have $|\langle x'm, f\rangle| = |\langle T'x', f\rangle| = |\langle x', T(f)\rangle| \leq q(T(f)) \leq ||T||_p^q$.

Thereby $(x'm)_p(X) = \|(x'm)_p\|_* = \|x'm\|_{p,\infty} \leq \|T\|_p^q$. Therefore, we have that $m_p^q(X) \leq \|T\|_p^q < \infty$. Also, $x'm \in M_p(X, E'_p)$, for each $x' \in F'$ such that $\|x'\|^{q,\infty} \leq 1$. To prove the condition (II), it is enough to prove that $(x'm)_p \in M_{\mathcal{P},t}(X)$, for each $x' \in F'$ such that $\|x'\|^{q,\infty} \leq 1$. Let $x' \in F'$ be such that $\|x'\|^{q,\infty} \leq 1$. By the condition (I), there exists $p_0 \in s(E)$ such that $x'm \in M_{\mathcal{P},p_0,t}(X, E'_{p_0})$. That is to say, $(x'm)_{p_0} \in M_{\mathcal{P},t}(X)$. By the lemma 3.2.13, we have that $(x'm)s \in M_{\mathcal{P},t}(X) \subset M_t(X)$. Then, by 3.2.12, we obtain $(x'm)_p \in M_t(X)$. Since T is (β_p, q)-continuous, given $\varepsilon > 0$, there exists a β_p-neighborhood of 0, we say V such that $\sup\{|\langle x', T(f)\rangle| : x' \in F', \|x'\|^{q,\infty} \leq 1\} = q(T(f)) < \varepsilon$, for each $f \in V$.

By 2.7.5, there exists $K \in \mathcal{P}$ such that $\{f \in C_b(X, E) : \|f\|_p \leq 1, \|f\|_{p,K} = 0\} \subset V$. Then, by the result 3.2.19, we have that $(x'm)_p \in M_{\mathcal{P},t}(X)$. Thus, we have finalized the proof of (II) and therefore $m \in M_{\mathcal{P},t}(X, \mathscr{L}(E, F))$.

<u>STEP 3:</u> We will prove that each $f \in C_b(X, E)$ is m-integrable and that $T(f) = \int_X f\, dm$.

Let $G \subset \mathcal{B}$ and $f \in C_b(X, E)$. For $x' \in F'$ we have

$$\langle x', j(T''(f\mathscr{X}_G))\rangle = \langle x', T''(f\mathscr{X}_G)\rangle = \langle T'x', f\mathscr{X}_G\rangle = \langle x'm, f\mathscr{X}_G\rangle = \int_G f\, d(x'm).$$

This proves that f is m-integrable and that $\int_G f\, dm = j(T''(f\mathscr{X}_G))$. Taking $G = X$, we obtain $\int_X f\, dm = j(T''(f\mathscr{X}_X)) = j(T''(i_0(f))) = T(f)$.

<u>STEP 4:</u> We will prove that $\|T\|_p^q = m_p^q(X)$, considering p and q previously mentioned.

For this, it remains to prove that $\|T\|_p^q \leq m_p^q(X)$. Let $f \in C_b(X, E)$ be with $\|f\|_p \leq 1$ and $x' \in F'$ be with $\|x'\|^{q,\infty} \leq 1$. We have that

$$|\langle x', T(f)\rangle| = |\langle T'x', f\rangle| = |\langle x'm, f\rangle| = \left|\int_X f\, d(x'm)\right| \leq \|x'm\|_{p,\infty} \leq m_p^q(X).$$

Thereby,

$$\sup\{|\langle x', T(f)\rangle| : x' \in F', \|x'\|^{q,\infty} \leq 1\} = q(T(f)) \leq m_p^q(X)$$

and therefore $\|T\|_p^q \leq m_p^q(X)$. Thus, we have proved (a).

<u>STEP 5:</u> We will prove that $m \in \mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E,F))$.

For this, it remains to prove that m satisfies the condition (R). Let $S \in Bd(E)$. Putting $A = \{f \in C_b(X,E) : f(X) \subset \overline{co}(S)\}$. We have that A is a τ_u-bounded set and therefore, is $\beta_{\mathcal{P}}$-bounded. Then, by hypothesis $T(A)$ is relatively $\sigma(F,F')$-compact. To satisfy the condition (R), it is enough to prove that $V_{m,S}$ is contained in the closure of $T(A)$ respect to the topology $\sigma(F,F')$.

Let $\{G_1, ..., G_n\}$ be a $\mathcal{B}$-partition of X and $s_1, ..., s_n \in S$. Let $\{x_j'\}_{i=1}^k \subset F'$. Let $q \in s(F)$. Considering $M > 0$ such that $\|x_j'\|^{q,\infty} \leq M$ for each $1 \leq j \leq k$. There exists $p \in s(E)$ such that $\frac{1}{M}x_j'm \in M_{\mathcal{P},p,t}(X, E_p')$, for each $1 \leq j \leq k$, and therefore $x_j'm \in M_{\mathcal{P},p,t}(X, E_p')$, for each $1 \leq j \leq k$. Since S is a bounded subset of E, we can define $d = \sup\{p(s) : s \in S\} < \infty$. Let a fix $j \in \{1, ..., k\}$. By the regularity of $(x_j'm)_p$, given $\varepsilon > 0$ there exist $Z_1, ..., Z_n \in \mathcal{Z}$, such that, $Z_i \subset G_i$ and $\sum_{i=1}^n (x_j'm)_p(G_i - Z_i) < \dfrac{\varepsilon}{2d}$.

Again, by regularity, there exist $U_1, ..., U_n \in \mathcal{U}$ pairwise disjoint such that $Z_i \subset U_i$ and $\sum_{i=1}^n (x_j'm)_p(U_i - Z_i) < \dfrac{\varepsilon}{2d}$. For each $i \in \{1, ..., n\}$, we choose $h_i \in C_b(x)$ such that $0 \leq h_i \leq \mathscr{X}_X$, $h_i^{-1}(\{1\}) = Z_i$ and $h_i^{-1}(\{0\}) = X - U_i$. Considering the function $h = \sum_{i=1}^n h_i \otimes s_i$, which belongs to A. Then $T(h) \in T(A)$. Also,

$$\left| x_j'\left(T(h) - \sum_{i=1}^n m(G_i)(s_i)\right)\right| = \left| x_j'\left(\sum_{i=1}^n \int_X h_i \otimes s_i \, dm - \sum_{i=1}^n m(G_i)(s_i)\right)\right|$$

$$
\begin{aligned}
&= \left| x'_j \left(\sum_{i=1}^{n} \int_{U_i} h_i \otimes s_i \, dm - \sum_{i=1}^{n} m(G_i)(s_i) \right) \right| \\
&= \left| x'_j \left(\sum_{i=1}^{n} \int_{Z_i} h_i \otimes s_i \, dm - \sum_{i=1}^{n} m(G_i)(s_i)) + \sum_{i=1}^{n} \int_{U_i \setminus Z_i} h_i \otimes s_i \, dm \right) \right| \\
&= \left| x'_j \left(\sum_{i=1}^{n} m(Z_i)(s_i) - \sum_{i=1}^{n} m(G_i)(s_i) + \sum_{i=1}^{n} \int_{U_i \setminus Z_i} h_i \otimes s_i \, dm \right) \right| \\
&\leq \sum_{i=1}^{n} \left| x'_j \left(m(G_i \setminus Z_i)(s_i) \right) \right| + \sum_{i=1}^{n} \left| \int_{U_i \setminus Z_i} h_i \otimes s_i \, d(x'_j m) \right| \\
&\leq \sum_{i=1}^{n} (x'_j m)_p (G_i \setminus Z_i) \cdot d + \sum_{i=1}^{n} (x'_j m)_p (U_i \setminus Z_i) \cdot d \\
&\leq \varepsilon,
\end{aligned}
$$

for any $j \in \{1, ..., m\}$. Therefore, $V_{m,S}$ is contained by the closure of $T(A)$ respect to the topology $\sigma(F, F')$.

STEP 6: We will prove the uniqueness of m.

Suppose that $m_1 \in \mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E,F))$ is a set function such that each $f \in C_b(X,E)$ is m_1-integrable and $T(f) = \int_X f \, dm_1$. Then for each $x' \in F'$, $\int_X f d(x'm) = \langle x', T(f) \rangle = \int_X f \, d(x'm_1)$, $\forall f \in C_b(X,E)$. Then, for each $x' \in F'$ we have $x'm = T'x' = x'm_1$ and from this $m = m_1$, since F is a locally convex Housdorff.

STEP 7: Let us prove the converse: if $m \in \mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E,F))$, then each $f \in C_b(X,E)$ is m-integrable and the operator $T : C_b(X,E) \to F$, defined by $T(f) = \int_X f \, dm$, is $(\beta_{\mathcal{P}}, \tau_F)$-continuous and $(\beta_{\mathcal{P}}, \tau_F)$-weakly compact.

Let $m \in \mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E,F))$. Consider $G \in \mathcal{B}$ and $f \in C_b(X,E)$. For $\alpha = \{G_1, \ldots, G_n, x_1, \ldots, x_n\} \in \Omega_G$, we have $\omega_\alpha(f,m) = \sum_{i=1}^{n} m(G_i)(f(x_i))$. If we put $S = f(X)$, then for each $\alpha \in \Omega_G$, $\omega_\alpha(f,m) \in V_{m,s}$.

By hypothesis, $V_{m,S}$ is relatively $\sigma(F,F')$-compact, then there exists a subnet $(\omega_{\alpha_\delta}(f,m))_{\delta \in \Lambda}$ of $(\omega_\alpha(f,m))_{\alpha \in \Omega_G}$ and a vector $z \in F$ such that for each $x' \in F'$, $\langle x', \omega_{\alpha_\delta}(f,m) \rangle \to \langle x', z \rangle$. On the other hand, for each $x' \in F'$, $x'm \in M_{\mathcal{P},p,t}(X, E'_p)$ and $(\langle x', \omega_\alpha(f,m) \rangle)_{\alpha \in \Omega_G} = (\omega_\alpha(f, x'm))_{\alpha \in \Omega_G}$ is convergent to $\int_X f \, d(x'm)$. Since $\mathbb{R}$ is Hausdorff, we have that $\langle x', z \rangle = \int_X f \, d(x'm)$. Therefore, each $f \in C_b(X,E)$ is m-integrable and $z = \int_X f \, dm$.

We define $T : C_b(X,E) \to F$ by $T(f) = \int_X f\, dm$.

The linear operator T is $(\beta_{\mathcal{P}}, \tau_F)$-continuous. In fact, given $q \in s(F)$, we choose $p \in s(E)$ such that $m_p^q(X) < \infty$. Let $\varepsilon > 0$ and $x' \in F'$ such that $\|x'\|^{q,\infty} \leq 1$. Then, there exists $r > 0$ such that $r \cdot (x'm)_p(X) < \frac{\varepsilon}{2}$ and $K \in \mathcal{P}$ such that, $(x'm)_p(G) < \frac{\varepsilon}{2}$ for each $G \subset X \backslash K$.

Let $f \in C_b(X,E)$ be such that $\|f\|_p \leq 1$ and $\|f\|_{p,K} \leq r$. Consider $G = \{x \in X : p(f(x)) \leq r\}$. We have that $G \in \mathcal{Z}$ and $X \backslash G \subset X \backslash K$. Then,

$$\begin{aligned} |\langle x', T(f)\rangle| =& \left| \int_X f\, d(x'm) \right| \\ \leq& \left| \int_G f\, d(x'm) \right| + \left| \int_{X\backslash G} f\, d(x'm) \right| \\ \leq& \sup_{x \in G}\{p(f(x))\} \cdot (x'm)_p(G) + \|f\|_p m_p(X\backslash G) \\ \leq& r \cdot (x'm)_p(X) + \frac{\varepsilon}{2} \\ \leq& \varepsilon \end{aligned}$$

Since $x' \in F'$, $\|x'\|^{q,\infty} \leq 1$, is arbitrary, it is concluded that $q(T(f)) \leq \varepsilon$. Thereby, we have proved that T is $\mathcal{P}_p^q$-tight. By 3.1.3, T is $(\beta_{\mathcal{P}}, \tau_F)$-continuous.

The operator T is $(\beta_{\mathcal{P}}, \tau_F)$-weakly compact. In fact, let $A \subset C_b(X,E)$ be a $\beta_{\mathcal{P}}$-bounded set. Then, the polar A^o is a $\beta((C_b(X,E), \beta_{\mathcal{P}})', (C_b(X,E), \beta_{\mathcal{P}}))$-neighborhood 0 in $(C_b(X,E), \beta_{\mathcal{P}})'$.

We define the bounded $S = \overline{co}(\bigcup\{f(X) : f \in A\})$. We will prove that $V_{m,S}$ is convex. Let $\alpha, \beta \geq 0, \alpha + \beta = 1$ and $\sum_{i=1}^n m(G_i)(s_i)$, $\sum_{j=1}^k m(H_i)(t_i) \in V_{m,S}$. Note that $\alpha s_i + \beta t_j \in S$. We have that:

$$\begin{aligned} \alpha \sum_{i=1}^n m(G_i)(s_i) + \beta \sum_{i=1}^k m(H_i)(t_i) &= \int_X \sum_{i=1}^n \alpha s_i \mathscr{X}_{G_i}\, dm + \int_X \sum_{j=1}^k \beta t_j \mathscr{X}_{H_i}\, dm \\ &= \int_X \sum_{i=1}^n \sum_{j=1}^k (\alpha s_i + \beta t_j) \mathscr{X}_{G_i \cap H_j}\, dm \\ &= \sum_{i=1}^n \sum_{j=1}^k m(G_i \cap H_j)(\alpha s_i + \beta t_j) \in V_{m,S} \end{aligned}$$

Now, we will prove that $V_{m,S}$ is balanced. Let $\lambda \in [-1,1]$ and $\sum_{i=1}^n m(G_i)(s_i) \in V_{m,S}$. We have that $\lambda(\sum_{i=1}^n m(G_i)(s_i)) = \sum_{i=1}^n m(G_i)(\lambda s_i) \in V_{m,S}$.

Thus, $V_{m,S}$ is absolutely convex and relatively $\sigma(F,F')$-compact, therefore, the polar $V_{m,S}^o$ is a $\tau(F',F)$-neighborhood of 0 in F'. We want to prove that T' is $(\tau(F',F), \beta((C_b(X,E),\beta_{\mathcal{P}})', (C_b(X,E),\beta_{\mathcal{P}})))$-continuous. For this, it is enough to prove that $T'(V_{m,S}^o) \subset A^o$. Let $x' \in V_{m,S}^o$ and $f \in A$. If $\{G_1,\ldots,G_n,x_1,\ldots,x_n\} \in \Omega_X$, then $\sum_{i=1}^n m(G_i)(f(x_i)) \in V_{m,S}$ and therefore $|\langle x', \sum_{i=1}^n m(G_i)(f(x_i))\rangle| \leq 1$.

This implies that $|\langle x', \int_X f\, dm\rangle| \leq 1$. Thereby $|\langle T'x', f\rangle| = |\langle x', T(f)\rangle| \leq 1$, which proves that $T'x' \in A^o$. The conclusion is obtained by applying the lemma 3.6.8. □

Definition 3.6.11. *We define the space*

$$\mathscr{L}_{\beta_{\mathcal{P}},w}(C_b(X,E),F) := \left\{ T : C_b(X,E) \to F : \begin{array}{c} T \text{ is a } (\beta_{\mathcal{P}},\tau_F)\text{-continuous and} \\ (\beta_{\mathcal{P}},\tau_F)\text{-weakly compact linear operator} \end{array} \right\}$$

This result can be reformulated as follows.

Theorem 3.6.12 (representation of continuous and weakly compact operators)**.**

The map $\Phi : \mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E,F)) \to \mathscr{L}_{\beta_{\mathcal{P}},w}(C_b(X,E),F)$, defined by $\Phi(m)(f) = \int_X f\, dm$, is bijective and is such that: $\forall x' \in F', x'm = \Phi(m)'x'$. Also, if $p \in s(E)$ and $q \in s(F)$ are such that $m_p^q(X) < \infty$, then $m_p^q(X) = \|\Phi(m)\|_p^q$.

Appendix A

Uniform structures.

In [3, 17] can be found the main definitions and properties of uniform spaces. This appendix is intended to state the results extracted from [3, 17] that are necessary for this document.

Let $\mathcal{X}$ be a set, G a covering of $\mathcal{X}$, a uniform space $(\mathcal{Y}, \mathcal{V})$ and $L(\mathcal{X}, \mathcal{Y})$ the space of functions from $\mathcal{X}$ to $\mathcal{Y}$. For each $V \in \mathcal{V}$, it is define

$$W(V) = \{(f, g) \in L(\mathcal{X}, \mathcal{Y}) \times L(\mathcal{X}, \mathcal{Y}) : \forall x \in \mathcal{X}, (f(x), g(x)) \in V\} .$$

The colection $\{W(V) : V \in \mathcal{V}\}$ forms a base for a uniformity on $L(\mathcal{X}, \mathcal{Y})$, denoted by $\mathscr{U}$. We say that $(L(\mathcal{X}, \mathcal{Y}), \mathscr{U})$ is the uniform structure of uniform convergence. The uniform structure of the G-convergence is the finest uniform structure on $L(\mathcal{X}, \mathcal{Y})$ such that for each $A \in G$, the map $R_A : L(\mathcal{X}, \mathcal{Y}) \to L(A, \mathcal{Y})$, $R_A(f) = f|_A$, is uniformly continuous.

The following result is shown in [3, theorem 2, page 30].

Theorem A.1. *(Ascoli Theorem)*

Let $\mathcal{X}$ be a topological space, G a covering of $\mathcal{X}$, a uniform space $(\mathcal{Y}, \mathcal{V})$ and a subset H of $L(\mathcal{X}, \mathcal{Y})$. Suppose that for each $A \in G$ and $f \in H$, the restriction $f|_A$ is continuous.

The H set is precompact respect to the uniform structure of the G-convergence, if and only if it verifies the following conditions:

(a) For each $A \in G$, the set $\{f|_A : f \in H\} \subset L(A, \mathcal{Y})$ is equicontinuous.

(b) For each $x \in \mathcal{X}$, the set $H(x) := \{f(x) : f \in H\} \subset \mathcal{Y}$ is precompact.

If $(\mathcal{X}, \mathscr{U})$ is any uniform structure, the topology in $\mathcal{X}$ induced by the uniformity $\mathscr{U}$, it will be denoted by $\tau_{\mathscr{U}}$.

The following result is shown in [17, theorem 6.7, page 46].

Theorem A.2. *Let $(\mathcal{X}, \mathscr{U})$ be a uniform structure. The space $(\mathcal{X}, \tau_{\mathscr{U}})$ is compact, if and only if $(\mathcal{X}, \mathscr{U})$ is complete and precompact.*

Appendix B

Convergence.

Let $\mathcal{X}$ be any topological space, D a subspace of $\mathcal{X}$ and $(\mathcal{E}, p)$ a seminormed space. We will say that a function $f : \mathcal{X} \to \mathcal{E}$ is ***p*-bounded in *D*** if

$$\|f\|_{p,D} := \sup_{x \in D} p(f(x)) < \infty.$$

Consider a net $(f_\alpha)_{\alpha \in \Lambda}$ of p-bounded functions in D. We will say that the net is ***p*-uniformly Cauchy** if, given $\varepsilon > 0$, there exists $\alpha_0 \in \Lambda$ such that $\|f_\alpha - f_\beta\|_{p,D} < \varepsilon$ for each pair $\alpha, \beta \geqslant \alpha_0$.

If there exists a function $f : \mathcal{X} \to \mathcal{E}$ such that for all $\varepsilon > 0$, there exists $\alpha_0 \in \Lambda$, for which $\|f_\alpha - f\|_{p,D} < \varepsilon$ for each $\alpha \geqslant \alpha_0$, then, the function f is p-bounded in D, and we will say that $(f_\alpha)_{\alpha \in \Lambda}$ is ***p*-uniformly convergent to f in *D***.

Proposition B.1. *Let $(f_\alpha)_{\alpha \in \Lambda}$ be a net of p-bounded functions in D and $f : \mathcal{X} \to \mathcal{E}$ a function. The following statements are equivalents:*

(a) The net $(f_\alpha)_{\alpha \in \Lambda}$ is p-uniformly Cauchy in D and pointwise convergent to f in D.

(b) The net $(f_\alpha)_{\alpha \in \Lambda}$ is p-uniformly convergent to f in D.

Proof. Is evidently that $(b) \Rightarrow (a)$. We will prove that $(a) \Rightarrow (b)$. Let $\varepsilon > 0$. By the condition p-uniformly Cauchy, there exists $\alpha_0 \in \Lambda$ such that for all $x \in D$,

$$\begin{aligned} p(f_\alpha(x) - f(x)) &\leqslant p(f_\alpha(x) - f_\beta(x)) + p(f_\beta(x) - f(x)) \\ &\leqslant \|f_\alpha - f_\beta\|_{p,D} + p(f_\beta(x) - f(x)) \\ &\leqslant \varepsilon + p(f_\beta(x) - f(x)) \ , \end{aligned} \tag{B.1}$$

for $\alpha, \beta \geqslant \alpha_0$. Consider a fix $y \in D$. By pointwise convergence, for $r > 0$, there exist $\beta_0 \in \Lambda$, $\beta_0 \geqslant \alpha_0$, such that: $p(f_{\beta_0}(y) - f(y)) \leqslant r$. Therefore, by (B.1) $p(f_\alpha(y) - f(y)) \leqslant \varepsilon + r$, where $r > 0$ is arbitrary, hence $p(f_\alpha(y) - f(y)) \leqslant \varepsilon$. Since $y \in D$ is arbitrary, it is concluded that $\|f_\alpha - f\|_{p,D} \leqslant \varepsilon$, for each $\alpha \geqslant \alpha_0$. □

Proposition B.2. *Let $(f_\alpha)_{\alpha\in\Lambda}$ be a net of continuous, p-bounded functions in D and $f : \mathcal{X} \to \mathcal{E}$ a function. If the net $(f_\alpha)_{\alpha\in\Lambda}$ is p-uniformly convergent to f in D, then, the restriction $f|_D$ is continuous, or in other words, $p \circ f|_D$ is continuous.*

Proof. Let $x_0 \in D$. Given $x \in D$ and $\alpha \in \Lambda$ we have that:

$$\begin{aligned} p(f(x) - f(x_0)) &\leqslant p(f(x) - f_\alpha(x)) + p(f_\alpha(x) - f_\alpha(x_0)) + p(f_\alpha(x_0) - f(x_0)) \\ &\leqslant \|f - f_\alpha\|_{p,D} + p(f_\alpha(x) - f_\alpha(x_0)) + \|f_\alpha - f\|_{p,D} \ . \end{aligned}$$

Given $\varepsilon > 0$, there exists $\alpha_0 \in \Lambda$ such that

$$p(f(x) - f(x_0)) \leqslant \frac{\varepsilon}{3} + p(f_{\alpha_0}(x) - f_{\alpha_0}(x_0)) + \frac{\varepsilon}{3} \ .$$

By the continuity of f_{α_0}, there exists a neighborhood V of x_0, such that, $p(f(x) - f(x_0)) \leqslant \varepsilon$, for each $x \in V \cap D$. Therefore, $f|_D$ is continuous in x_0. □

Appendix C

Barreled, quasibarreled, bornological, ultrabornological, angelic and (DF)-spaces.

The classical definitions and the main properties of these spaces, except the ultrabornological, angelic and (DF)-spaces, can be found in [26].

One of the purposes of this appendix is to prove equivalences between the definitions that will be used in this document and the definitions given in [26]. For an alternative definition of barreled spaces, it was used [4], and from this, it was deduced an alternative definition for quasibarreled spaces. In the case of the (DF)-spaces, we will follow the definition given by Grothendieck in [10]. In the literature there was not found the necessary equivalence between the definition of a ultrabornological space, which Schmets used in [21], and the proposed in [10]. However, it was proved an equivalence of, at least, 22 definitions for such space. All what we need from angelic spaces, is found in [18] and we will collect the results used in this document.

C.1. Barreled and quasibarreled spaces.

Definition C.1.1. *Let $\mathcal{X}$ be a topological space. A function $f : \mathcal{X} \to \mathbb{R}$ is* ***lower semi-continuous at a in $\mathcal{X}$****, if for each $h \in \mathbb{R}$ such that $h < f(a)$, there exists a neighborhood U of a such that $h < f(x)$ for $x \in U$. If f is lower semi-continuous for all $x \in \mathcal{X}$, we say that is* ***lower semi-continuous in $\mathcal{X}$****.*

Lemma C.1.2. *For a function $f : \mathcal{X} \to \mathbb{R}$ to be lower semi-continuous in $\mathcal{X}$, is necessary and sufficient that for each $k \in \mathbb{R}$, the set $f^{-1}(k, +\infty)$ is open.*

Proof. Suppose that for each $k \in \mathbb{R}$, the set $f^{-1}(k, +\infty)$ is open in $\mathcal{X}$. Let $a \in \mathcal{X}$ and $h \in \mathbb{R}$, $h < f(a)$. Since $f^{-1}(k, +\infty)$ is open in $\mathcal{X}$, we have the lower semi-continuity of f in a. Now suppose that f is lower semi-continuous in $\mathcal{X}$ and that there exists $k \in \mathbb{R}$ such that the set $H := f^{-1}(k, +\infty)$ is not open. Therefore, $H \neq \emptyset$. Given $a \in \mathcal{X}$ such that $k < f(a)$, there exists a neighborhood U of a such that $U \subset H$, hence H is a neighborhood of all of its elements, which is a contradiction. □

Lemma C.1.3. *Let $\{f_\alpha : \alpha \in \Lambda\}$ be a family of lower semi-continuous functions at $a \in \mathcal{X}$. If we can define the function $g : \mathcal{X} \to \mathbb{R}$, $g(x) := \sup\{f_\alpha(x) : \alpha \in \Lambda\}$, then, is lower semi-continuous in a.*

Proof. Let $h \in \mathbb{R}$ be such that $h < g(a)$. There exists $\alpha \in \Lambda$ such that $f < f_\alpha(a) \leqslant g(a)$.

Then, there exists a neighborhood U of a such that $h < f_\alpha(x) \leqslant g(x)$ for each $x \in U$. □

Definition C.1.4. *Let E be a Hausdorff locally convex space and B a subset of E. We say that B is a* ***bornivorous****, if and only if absorbs all bounded subsets of E. We say that B is a* ***barrel****, if and only if is absolutely convex, closed and absorbent.*

Proposition C.1.5. *Let B be a subset of E. The following statements are equivalents:*

(a) B is a barrel and a bornivorous.

(b) There exists a subset M of E', absolutely convex and $\beta(E', E)$-bounded such that $M_o := \{e \in E : |\langle y, e\rangle| \leqslant 1, \forall y \in M\} = B$.

(c) *There exists a seminorm p lower semi-continuous in E, bounded in bounded subsets of E such that $B = \{e \in E : p(e) \leqslant 1\}$.*

Proof. $(a) \Rightarrow (b)$: If B is a barrel and a bornivorous, then, the set $M = B^o$ is $\beta(E', E)$-bounded and absolutely convex, this by [26, proposition 9, page 246].

$(b) \Rightarrow (c)$: Let M be a subset of E', absolutely convex and $\beta(E', E)$-bounded, such that $B = M_o$. Then, we can define the seminorm p in each $e \in E$, as follows: $p(e) := \sup\{|f(e)| : f \in M\}$. By the previous lemma, p is lower semi-continuous in X. Let A be a bounded subset of E. Since M is $\beta(E', E)$-bounded, we have that $\sup\{f(e) : e \in A, f \in M\} < \infty$. That is to say, p is bounded in A. Also, $B = M_o = \{e \in E : |f(e)| \leqslant 1, \forall f \in M\} = \{e \in E : p(e) \leqslant 1\}$.

$(c) \Rightarrow (a)$: Let p be a seminorm lower semi-continuous in E, bounded in bounded subsets of E, such that $B = \{e \in E : p(e) \leqslant 1\}$. Clearly B is absolutely convex and absorbent. Since $E \backslash B = p^{-1}(1, \infty)$ is open (C.1.2), it has been completed to prove that B is a barrel. Let A be a bounded subset of E. We define $L = \sup\{p(e) : e \in A\}$. Then, $A \subset LB$, proving that B is a bornivorous. □

Corollary C.1.6. *The following statements are equivalents:*

(a) *Each $\beta(E', E)$-bounded set is equicontinuous.*

(b) *Each bornivorous barrel of E is neighborhood of* 0.

(c) *Each seminorm lower semi-continuous in E, bounded in each bounded subset of E, is continuous.*

Proof. $(a) \Rightarrow (b)$: Suppose that (a) is satisfied. Let B be a bornivorous barrel of E. By the previous proposition, there exists a $\beta(E', E)$-bounded, absolutely convex M such that $M_o = B$. Since M is equicontinuous, we have that B is a neighborhood of 0.

$(b) \Rightarrow (c)$: Now suppose that (b) is satisfied. Let p be a seminorm lower semi-continuous in E, bounded in each bounded subset of E. By the previous proposition, the set $\{e \in E : p(e) \leqslant 1\}$ is barrel bornivorous, then, the continuity is obtained by (b).

$(c) \Rightarrow (a)$: Finally, suppose that (c) is satisfied. Let B be a $\beta(E', E)$-bounded set. The convex hull of B, $co(B)$, is also a $\beta(E', E)$-bounded set.

In fact, let A be a bounded subset of E, $L = \sup\{f(e) : e \in A, f \in M\}$, $f_1, \ldots, f_n \in B$, $t_1, \ldots, t_n \in \mathbb{K}$, such that $\sum_{i=1}^{n} |t_i| \leqslant 1$ and $e \in E$.

$$|\sum_{i=1}^{n} t_i f_i(e)| \leqslant \sum_{i=1}^{n} |t_i||f_i(e)| \leqslant \text{máx}\{|f_i(e)| : i = 1, \ldots, n\} \leqslant L$$

Applying the previous proposition, there exists a seminorm p lower semi-continuous in E, bounded in each bounded subset of E such that

$$(co(B))_o = \{e \in E : p(e) \leqslant 1\} \ .$$

By hypothesis, p is continuous and from this $(co(B))_o$ is a neighborhood of 0. Applying [26, theorem 6, page 233] it is obtained that $co(B)$ is equicontinuous and therefore, so is B. □

Definition C.1.7. *A Hausdorff locally convex space is a **quasibarreled space** if and only if it satisfies any of the equivalent statements of the corollary C.1.6.*

Proposition C.1.8. *Let B be a subset of E. The following statements are equivalents:*

(a) B is a barrel.

(b) There exists a subset M of E', absolutely convex and $\sigma(E', E)$-bounded such that $M_o = B$.

(c) There exists a seminorm p lower semi-continuous in E such that $B = \{e \in E : p(e) \leqslant 1\}$.

Proof. $(a) \Rightarrow (b)$: If B is a barrel, then the set $M = B^o$ is $\sigma(E', E)$-bounded and absolutely convex, this by [26, proposition 4, page 296].

$(b) \Rightarrow (c)$: Let M be a subset of E', absolutely convex and $\sigma(E', E)$-bounded such that $B = M_o$. Then, we can define the seminorm p in each $e \in E$, as follows: $p(e) := \sup\{|\langle f, e\rangle| : f \in M\}$. By the lemma C.1.3, p is lower semi-continuous in X. Is clear that $B = M_o = \{e \in E : |f(e)| \leqslant 1, \forall f \in M\} = \{e \in E : p(e) \leqslant 1\}$.

$(c) \Rightarrow (a)$: Let p be a seminorm lower semi-continuous in E, such that $B = \{e \in E : p(e) \leqslant 1\}$. Then, following the same argument of C.1.5, B is a barrel. □

Corollary C.1.9. *The following statements are equivalents:*

(a) Each $\sigma(E', E)$-bounded set is equicontinuous.

(b) Each barrel of E is neighborhood of 0.

(c) Each seminorm lower semi-continuous in E, is continuous.

Proof. $(a) \Rightarrow (b)$: Suppose that (a) is satisfied. Let B be a barrel of E. By the previous proposition, there exists a $\sigma(E', E)$-bounded, absolutely convex M such that $M_o = B$. Since M is equicontinuous, we have that B is neighborhood of 0.

$(b) \Rightarrow (c)$: Now suppose that (b) is satisfied. Let p be a seminorm lower semi-continuous in E. By the previous proposition, the set $\{e \in E : p(e) \leqslant 1\}$ is barrel, then, the continuity is obtained by (b).

$(c) \Rightarrow (a)$: Finally, suppose that (c) is satisfied. Let B be a $\sigma(E', E)$-bounded set. The convex hull of B, $co(B)$, is also a $\sigma(E', E)$-bounded set.

In fact, let $e \in E$, $L = \sup\{f(e) : f \in M\}$, $f_1, \ldots, f_n \in B$, $t_1, \ldots, t_n \in \mathbb{K}$ be such that $\sum_{i=1}^{n} |t_i| \leqslant 1$.

$$|\sum_{i=1}^{n} t_i f_i(e)| \leqslant \sum_{i=1}^{n} |t_i||f_i(e)| \leqslant \text{máx}\{|f_i(e)| : i = 1, \ldots, n\} \leqslant L$$

Then, by the previous proposition, there exists a seminorm p lower semi-continuous in E such that

$$(co(B))_o = \{e \in E : p(e) \leqslant 1\} \ .$$

By hypothesis, p is continuous and from this $(co(B))_o$ is a neighborhood of 0. Applying [26, theorem 6, page 233] it is obtained that $co(B)$ is equicontinuous and therefore, so is B. □

Definition C.1.10. *A Hausdorff locally convex space is a* ***barreled space*** *if and only if it satisfies any of the equivalent statements of the corollary C.1.9.*

C.2. (DF)-spaces.

Definition C.2.1. *Let E be a Hausdorff locally convex space. A family $\mathscr{B}$ of bounded subsets of E, is a* ***fundamental system of bounded sets*** *if every bounded subset of E is contained in some element of $\mathscr{B}$.*

Definition C.2.2. *A Hausdorff locally convex space E is a (DF)-space if and only if it satisfies the following conditions:*

(a) E admits a fundamental system of countable bounded sets.

(b) Each sequence $(U_n)_{n\in\mathbb{N}}$ of absolutely convex and closed 0-neighborhood, whose intersection is a bornivorous subset, this intersection is a 0-neighborhood.

Proposition C.2.3. *The following statements are equivalents:*

(a) E admits a fundamental system of countable bounded sets.

(b) $(E', \beta(E', E))$ is metrizable.

Proof. $(a) \Rightarrow (b)$: If $\mathscr{B}$ is a fundamental system of bounded sets in E countable, then $\{B^o : B \in \mathscr{B}\}$ is a countable neighborhood base of $(E', \beta(E', E))$. Then, by the Kakutani Theorem ([26, Theorem 3, page 33]), $(E', \beta(E', E))$ is metrizable.

$(b) \Rightarrow (a)$: If $(E', \beta(E', E))$ is metrizable, then we have a neighborhood base of 0 countable $\{B_n^o : n \in \mathbb{N}\}$, where (B_n) is a sequence of bounded subsets of E, which is a fundamental system of bounded. □

Lemma C.2.4. *A subset B of E' is $\beta(E', E)$-bounded if and only if B_o is a bornivorous.*

Proof. Let A be a bounded subset of E and $n \in \mathbb{N}$. The result follows the following equivalence:

$$\begin{aligned}\sup\{|f(e)| : e \in A, f \in B\} \leqslant n &\Leftrightarrow \sup\{|f(\frac{1}{n}e)| : e \in A, f \in B\} \leqslant 1\\ &\Leftrightarrow \frac{1}{n}A \subset B_o\end{aligned}$$

□

Proposition C.2.5. *The following statements are equivalents:*

(a) Each sequence $(U_n)_{n\in\mathbb{N}}$ of absolutely convex and closed 0-neighborhood, whose intersection is a bornivorous subset, this intersection is a 0-neighborhood.

(b) Each $\beta(E', E)$-bounded which is countable union of equicontinuous sets, is equicontinuous.

(c) Every sequence $(p_n)_{n\in\mathbb{N}}$ of continuous seminorms in E such that $p = \sup_{n\in\mathbb{N}} p_n$ is a seminorm in E, bounded in each bounded subset of E, satisfies that p is continuous.

Proof. $(a) \Rightarrow (b)$: Let H be a $\beta(E', E)$-bounded subset and suppose that $\{H_n : n \in \mathbb{N}\}$ is a colection of equicontinuous subsets of E' such that $\bigcup_{n=1}^{\infty} H_n = H$. We have $H_o = \bigcap_{n=1}^{\infty}(H_n)_o$. Then, $((H_n)_o)$ is a sequence of neighborhoods of 0, absolutely convex and closed. By the previous lemma and by (a), we have that H_o is a neighborhood of 0. Therefore $(H_o)^o = H$ is equicontinuous.

$(b) \Rightarrow (a)$: Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of neighborhoods of 0, absolutely convex and closed, whose intersection U, is a bornivorous. Then, $U^o = \bigcup_{n=1}^{\infty}(U_n)^o$, where $(U_n)^o$ is equicontinuous for each $n \in \mathbb{N}$. By the previous lemma and by (b), we have that $(U^o)_o = U$ is a neighborhood of 0.

$(a) \Rightarrow (c)$: Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of continuous seminorms in E such that $p = \sup_{n\in\mathbb{N}} p_n$ is a seminorm in E, bounded in each bounded subset of E. Defining $K = \{e \in E : p(x) \leqslant 1\}$ and $K_n = \{e \in E : p_n(x) \leqslant 1\}$ for each $n \in \mathbb{N}$. Thus, $K = \bigcap_{n=1}^{\infty} K_n$, where K_n is a closed neighborhood of 0, absolutely convex for each $n \in \mathbb{N}$. Let A be a bounded subset of E and defining $L = \sup\{p(e) : e \in A\}$. Then, $A \subset LK$. Thereby, K is a bornivorous and by hypothesis K is neighborhood of 0.

$(c) \Rightarrow (a)$: Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of neighborhoods of 0, absolutely convex and closed, whose intersection U, is a bornivorous. We define for each $n \in \mathbb{N}$, p_n the Minskowski functional in U_n Thereby, for each $n \in \mathbb{N}$, we have that p_n is continuous in E and $U_n = \{e \in E : p_n(e) \leqslant 1\}$. If we denote by p the Minskowski functional in U, then $U = \{e \in E : p(e) \leqslant 1\} = \{e \in E : \sup_{n\in\mathbb{N}} p_n(e) \leqslant 1\}$.

Then, $p(e) = 0$ if and only if $\sup_{n\in\mathbb{N}} p_n(e) = 0$. If $p(e) \neq 0$, then $e/p(e) \in U$ and from this $\sup_{n\in\mathbb{N}} p_n(e) \leqslant p(e)$. On the other hand, $e/\sup_{n\in\mathbb{N}} p_n(e) \in U$, and from this $p(e) \leqslant \sup_{n\in\mathbb{N}} p_n(e)$. Summarizing, $p = \sup_{n\in\mathbb{N}} p_n$, and since U is a bornivorous, it follows that p is bounded. Applying (c), it is concluded that p is continuous, and thereby U is a neighborhood 0. □

From the propositions C.2.3 and C.2.5 it follows that there exist, at least, 6 equivalents definitions for a (DF)-space. The definition that we choose is the following:

Definition C.2.6. *A Hausdorff locally convex space E is a (DF)-space if and only if it satisfies the following conditions:*

(a) E admits a fundamental system of countable bounded sets.

(b) Ever sequence $(p_n)_{n\in\mathbb{N}}$ of continuous seminorms in E such that $p = \sup_{n\in\mathbb{N}} p_n$ is a seminorm in E, bounded in each bounded subset of E, satisfies that p is continuous.

C.3. Bornological and ultrabornological spaces.

Definition C.3.1. *A Hausdorff locally convex space is **bornological** if each absolutely convex bornivorous subset is a neighborhood of* 0.

Let E be a Hausdorff locally convex space and let B be a bounded subset and absolutely convex of E. Putting $\langle B\rangle = \operatorname{span}(B) = \bigcup_{n=1}^{\infty} nB$ and noting that B is an absolutely convex subset and absorbent of $\langle B\rangle$. Then, the Minkowski functional p_B of B is a seminorm in $\langle B\rangle$. In fact, p_B is a norm in $\langle B\rangle$. Indeed, let $e \in E$ be such that $p_B(e) = 0$. Then, for each $n \in \mathbb{N}$, there exists $\alpha > 0$ such that $\alpha < \frac{1}{n}$ and $e \in \alpha B \subset \frac{1}{n}B$. If $e \neq 0$, then (ne) is a sequence in B such that $(\frac{1}{n}ne)$ is not convergent to 0. Therefore, B is not bounded, which is a contradiction.

Let $\omega_B = \sigma(\langle B\rangle, \{p_B\})$ be the topology in $\langle B\rangle$ induced by the norm p_B. It is stated that the subset topology in $\langle B\rangle$ induced by τ_E is less fine than ω_B. In fact, note that

$\{e \in E : p_B(e) < 1\} \subset B$ and given a $\tau_{_E}$-neighborhood U of 0, there exists $t > 0$ such that $B \subset tU$. Then, $\{e \in E : p_B(e) < \frac{1}{t}\} \subset \frac{1}{t}B \subset U \cap \langle B\rangle$. Also, it follows that the canonic injection $j_B : \langle B\rangle \to E$ is $(\omega_B, \tau_{_E})$-continuous.

Let $\mathscr{B}$ be the family of all the subsets of E that are simultaneously bounded and absolutely convex. If $\mathscr{A} \subset \mathscr{B}$, then, the inductive limit produced by the family $\{(j_B, \langle B\rangle) : B \in \mathscr{A}\}$ will be denoted by $\mathrm{ind}_{B\in\mathscr{A}}(j_B, \langle B\rangle)$.

Theorem C.3.2. *Let E be a Hausdorff locally convex space. The following statements are equivalents:*

(a) E is bornological.

(b) Each seminorm in E, bounded in each bounded subset of E, is continuous.

(c) Given F a Hausdorff locally convex space and $T : E \to F$ a bounded linar map, we have that T is continuous.

(d) E is an inductive limit of normed spaces.

(e) $(E, \tau_{_E}) = \mathrm{ind}_{B\in\mathscr{B}}(j_B, \langle B\rangle)$.

Proof. Immediate of [26, Theorem 1, page 261],[26, Theorem 3, page 269] and from [26, Exercise 1, page 274] . □

Definition C.3.3. *A Hausdorff locally convex space is* ***ultrabonological*** *if and only if it is an inductive limit of Banach spaces.*

Definition C.3.4. *Let B be a bounded and absolutely convex subset of E. We say that B is* ***completant*** *if and only if the space $(\langle B\rangle, \omega_B)$ is Banach.*

Lemma C.3.5. *Let E be a Hausdorff locally convex space and $B \subset E$ a closed, bounded and absolutely convex. Suppose that (e_n) is a Cauchy sequence in $\langle B\rangle$. We have that, (e_n) is ω_B-convergent in $\langle B\rangle$, if and only if, it is $\tau_{_E}$-convergent in E.*

Proof. Suppose that (e_n) is ω_B-convergent in $\langle B\rangle$. Since the canonic injection j_B is continuous, we have that (e_n) is $\tau_{_E}$-convergent in E. Now suppose that (e_n) is $\tau_{_E}$-convergent in E. There exists an increasing sequence of integers (n_k) such that: $p_B(e_m - e_n) < \frac{1}{2^{k+1}}$, for each $m, n \geqslant n_k$. Since B is closed, we can define the following decreasing sequence of closed sets $(e_{n_k} + \frac{1}{2^k}B)$. By the previous inequality, given $k \in \mathbb{N}$, $e_m \in e_{n_k} + \frac{1}{2^k}B$, for each $m \geqslant n_k$. Then, the convergence of (e_m) implies that $y = \text{lím}\, e_m \in e_{n_k} + \frac{1}{2^k}B$, for each $k \in \mathbb{N}$. Thus $y \in \bigcap_{n=1}^{\infty} e_{n_k} + \frac{1}{2^k}B$ and from this (e_{n_k}) is ω_B-convergent to y in $\langle B\rangle$. □

Lemma C.3.6. *If $B \subset E$ is a bounded, absolutely convex and sequentially complete subspace, then B is completant.*

Proof. Let (e_n) be a Cauchy sequence in $\langle B\rangle$. Then, there exists $m \in \mathbb{N}$ such that $\{e_n : n \in \mathbb{N}\} \subset mB$. Note that mB is also sequentially complete. Therefore, (e_n) is $\tau_{_E}$-convergent to an element y of mB. Applying the previous lemma, (e_n) is ω_B-convergent. Therefore, the space $(\langle B\rangle, \omega_B)$ is Banach. □

Consequently, we have the following:

Corollary C.3.7. *Let $B \subset E$ be a bounded and absolutely convex subspace. If B is complete or compact, then is completant.*

Now we will show a series of equivalent definitions for an ultrabornological space.

Theorem C.3.8. *Let E be a Hausdorff locally convex space. If $\mathcal{K}$ is the family of all the subsets of E, that are absolutely convex and simultaneously compacts, $\mathcal{C}$ the family of all the completants subsets of E and $\mathcal{V}$ is a faimily of subsets of E, such that $\mathcal{K} \subset \mathcal{V} \subset \mathcal{C}$, then the following statements are equivalents:*

(u) E is ultrabornological.

(c1) Every seminorm in E, bounded in each completant set, is continuous.

(v1) Every seminorm in E, bounded in each element of $\mathcal{V}$, is continuous.

(k1) *Every seminorm in E, bounded in each absolutely convex and compact set, is continuous.*

(c2) *Every absolutely convex subset of E, which absorbs each completant subset of E, is neighborhood of* 0.

(v2) *Every absolutely convex subset of E, which absorbs each element of $\mathcal{V}$, is neighborhood of* 0.

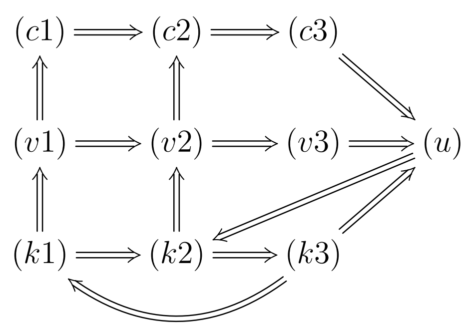
(k2) *Every absolutely convex subset of E, which absorbs each absolutely convex and compact subset of E, is neighborhood of* 0.

(c3) $(E, \tau_{E}) = \mathrm{ind}_{B \in \mathcal{C}}(j_B, \langle B \rangle)$.

(v3) $(E, \tau_{E}) = \mathrm{ind}_{B \in \mathcal{V}}(j_B, \langle B \rangle)$.

(k3) $(E, \tau_{E}) = \mathrm{ind}_{B \in \mathcal{K}}(j_B, \langle B \rangle)$.

Proof. We will prove the theorem proving the implications of the following diagram:



Is immediate that: $(k3) \Rightarrow (u)$, $(v3) \Rightarrow (u)$ and that $(c3) \Rightarrow (u)$. Since $\mathcal{K} \subset \mathcal{V} \subset \mathcal{C}$, is also immediate that $(k1) \Rightarrow (v1) \Rightarrow (c1)$ and $(k2) \Rightarrow (v2) \Rightarrow (c2)$.

Suppose that (c1) (respectively (v1))(respectively (k1)) is satisfied.

Let H be an absolutely convex subset of E which absorbs each element of $\mathcal{C}$ (resp. $\mathcal{V}$)(resp. $\mathcal{K}$). Note that H is absorbent. Then, we can define p_H the Minkowski functional of H. Let $B \in \mathcal{C}$ (resp. $\mathcal{V}$)(resp. $\mathcal{K}$). There exists $r > 0$ such that

$B \subset rH \subset \{e \in E : p_H(e) \leqslant r\}$. Thus, p_H is bounded in B and by hypothesis, is continuous.

Thereby H is a neighborhood of 0, since $\{e \in E : p_H(e) < 1\} \subset H$. Thus we have proved $(c1) \Rightarrow (c2)$ (resp. $(v1) \Rightarrow (v2)$)(resp. $(k1) \Rightarrow (k2)$)

Now suppose that (c2) (resp. (v2))(resp. (k2)) is satisfied. We have that the collection $\mathscr{U}_{\mathcal{C}}$ (resp. $\mathscr{U}_{\mathcal{V}}$)(resp. $\mathscr{U}_{\mathcal{K}}$) of all the absolutely convex subsets of E, which absorbs each element of $\mathcal{C}$ (resp. $\mathcal{V}$)(resp. $\mathcal{K}$), is a base of τ_E-neighborhood of 0. Consider the space $(E, \tau_{\mathcal{C}}) := \operatorname{ind}_{B \in \mathcal{C}}(j_B, \langle B \rangle)$ (resp. $(E, \tau_{\mathcal{V}}) := \operatorname{ind}_{B \in \mathcal{V}}(j_B, \langle B \rangle))$ (resp. $(E, \tau_{\mathcal{K}}) := \operatorname{ind}_{B \in \mathcal{K}}(j_B, \langle B \rangle))$.

Recall that the collection $\mathscr{U}^*_{\mathcal{C}}$ $(\mathscr{U}^*_{\mathcal{V}})(\mathscr{U}^*_{\mathcal{K}})$ defined by

$$\left\{ U \subset E : \begin{array}{c} U \text{ is absolutely convex such that } j_B^{-1}(U) \\ \text{is a } \omega_B\text{-neighborhood of 0 in } \langle B \rangle, \forall B \in \mathcal{C}(\mathcal{V})(\mathcal{K}) \end{array} \right\}$$

is a base of $\tau_{\mathcal{C}}(\tau_{\mathcal{V}})(\tau_{\mathcal{K}})$-neighborhood of 0 in E.

We will prove that $\tau_E \leqslant \tau_{\mathcal{C}}$ (resp. $\tau_E \leqslant \tau_{\mathcal{V}}$)(resp. $\tau_E \leqslant \tau_{\mathcal{K}}$). Let $H \in \mathscr{U}_{\mathcal{C}}$ $(\mathscr{U}_{\mathcal{V}})(\mathscr{U}_{\mathcal{K}})$.

For $B \in \mathcal{C}(\mathcal{V})(\mathcal{K})$, there exists $r > 0$ such that $B \subset rH$. Thus, $B \subset rH \cap \langle B \rangle$. Therefore $rH \cap \langle B \rangle$ is a ω_B-neighborhood of 0 in $\langle B \rangle$, and from this, so is $j_B^{-1}(H) = H \cap \langle B \rangle$. Thereby $H \in \mathscr{U}^*_{\mathcal{C}}$ $(\mathscr{U}^*_{\mathcal{V}})(\mathscr{U}^*_{\mathcal{K}})$.

Now we will prove that $\tau_{\mathcal{C}} \leqslant \tau_E$ (resp. $\tau_{\mathcal{V}} \leqslant \tau_E$)(resp. $\tau_{\mathcal{K}} \leqslant \tau_E$). Let $H \in \mathscr{U}^*_{\mathcal{C}}$ $(\mathscr{U}^*_{\mathcal{V}})(\mathscr{U}^*_{\mathcal{K}})$. Since each $B \in \mathcal{C}(\mathcal{V})(\mathcal{K})$ is ω_B-bounded in $\langle B \rangle$, there exists $r > 0$ such that $B \subset r(U \cap \langle B \rangle) = rU \cap \langle B \rangle \subset rU$. Therefore, $U \in \mathscr{U}_{\mathcal{C}}$ $(\mathscr{U}_{\mathcal{V}})(\mathscr{U}_{\mathcal{K}})$. Thus, we have proved that (c3)(resp. (v3))(resp. (k3)) is satisfied.

Now let us prove that $(k3) \Rightarrow (k1)$. Suppose that (k3) is satisfied. Let q be a seminorm in E, bounded in each element of $\mathcal{K}$. Consider the set $U = \{e \in E : q(e) \leqslant 1\}$. Given $B \in \mathcal{C}$, there exists $r > 0$ such that $B \subset rU$. For each $e \in \langle B \rangle$, we have that

$$\{c > 0 : e \in cB\} \subset \{c > 0 : e \in crU\}.$$

Then, for all $e \in \langle B \rangle$, $\frac{1}{r}q(e) = \frac{1}{r}p_U(e) \leqslant p_B(e)$, where p_U and p_B are the Minkowski functionals of U and B respectively, both defined in $\langle B \rangle$. Thereby, $\frac{1}{r}q \circ j_B \leqslant p_B$ en $\langle B \rangle$, hence $\frac{1}{r}q \circ j_B$ is ω_B-continuous in $\langle B \rangle$.

Recall that $\tau_{\mathcal{K}} = \sigma(E, P)$, where P is the family of seminorms in E defined by $\{p : p \circ j_B$ is ω_B-continuous in $\langle B\rangle, \forall B \in \mathcal{K}\}$. Therefore q is $\tau_{\mathcal{K}}$-continuous and by hypothesis, is $\tau_{_E}$-continuous. Thus we have proved that (k1) is satisfied.

Now it is enough to prove that $(u) \Rightarrow (k2)$. Suppose that $(E, \tau_{_E}) = \mathrm{ind}_{\alpha\in\Lambda}(T_\alpha, E_\alpha)$, where $(E_\alpha)_{\alpha\in\Lambda}$ is a family of Banach spaces and for each $\alpha \in \Lambda$, T_α is a linear map of E_α to E. Let $V \in \mathscr{U}_{\mathcal{K}}$. Given $\alpha \in \Lambda$ and B_α a closed ball of E_α, V absorbs $T_\alpha(B_\alpha)$. Otherwise, there exists $\alpha \in \Lambda$ such that for each $n \in \mathbb{N}$, there exists $x_n \in B_\alpha$ such that $T_\alpha(x_n) \notin n^2V$, this is, $T_\alpha(\frac{1}{n}x_n) \notin nV$. Since B_α is bounded in E_α, we have the sequence $(\frac{1}{n}x_n)$ is convergent to 0. Then, the set $\{0\} \cup \{\frac{1}{n}x_n : n \in \mathbb{N}\}$ is compact in E_α. Since the set $K = \overline{co(\{0\} \cup \{\frac{1}{n}x_n : n \in \mathbb{N}\})}$ is precompact and the space E_α is complete, by **??** it is concluded that K is absolutely convex compact. By the continuity and linearity of T_α we have that $T_\alpha(K)$ is a absolutely convex compact subset of E and therefore is absorbed by V. That is to say, there exists $r > 0$, such that $T_\alpha(\frac{1}{n}x_n) \in rV$, for all $n \in \mathbb{N}$. We choose $n > r$ and we obtain that $T_\alpha(\frac{1}{n}x_n) \in nV$, which is a contradiction. Thereby, for all $\alpha \in \Lambda$, there exists $r > 0$ such that $T_\alpha(B_\alpha) \subset rV$. Thus, $\frac{1}{r}B_\alpha \subset T_\alpha^{-1}(V)$. Therefore, $T_\alpha^{-1}(V)$ is a neighborhood of 0 in E_α, for each $\alpha \in \Lambda$. Thus V is neighborhood of 0 in $(E, \tau_{_E}) = \mathrm{ind}_{\alpha\in\Lambda}(T_\alpha, E_\alpha)$. □

Now, to obtain at least 22 equivalences in the previous theorem, it is enough to make explicit the family $\mathcal{V}$. For example, we can consider $\mathcal{V}_1(\mathcal{V}_2)(\mathcal{V}_2')(\mathcal{V}_3)(\mathcal{V}_3')$ as the family of all the subsets of E, that are bounded, absolutely convex and completes (resp. sequentially completes and closed)(resp. sequentially completes, closed and precompacts)(resp. sequentially completes)(resp. sequentially completes and precompacts) simultaneously. Thereby, $\mathcal{K} \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{V}_3 \subset \mathcal{C}$ and $\mathcal{K} \subset \mathcal{V}_1 \subset \mathcal{V}_2' \subset \mathcal{V}_3' \subset \mathcal{C}$.

C.4. Angelical Spaces.

A subspace A of a topological space S is:

- **relatively countable compact (resp. countable compact)** if each sequence in A have an accumulation point in S (resp. in A).
- **relatively sequentially compact (resp. sequentially compact)** if each sequence in A have a sequence convergent to a point in S (resp. in A).
- **σ-compact** if this is the countable union of compact subspaces.

The following definition is due to D.H. Fremlin.

Definition C.4.1. *A topological Hausdorff space S is an* ***angelical space*** *if it satisfies the following three properties:*

(a) For any subset A of S are equivalents:

(i) A is relatively compact.

(ii) A is relatively countably compact.

(iii) A is relatively sequentially compact.

(b) Similarly are equivalent:

(i) A is compact.

(ii) A is countably compact.

(iii) A is sequentially compact.

(c) If A is a relatively compact subspace, each point in $\overline{A}$ is the limit of a sequence in A.

In [18] are shown the proofs of the following propositions:

Proposition C.4.2. *If there exists a continuous and injective function f from a regular space S in an angelical space T, then S is also angelical.*

Proposition C.4.3. *If T have a σ-compact dense subspace, then $(C(T), pw)$ is angelical.*

Proposition C.4.4. *If $(C(T), pw)$ is angelical, then $(C(T, Z), pw)$ is angelical for any metrical space Z.*

Corollary C.4.5. *If T have a σ-compact dense subspace and Z is metrizable, then $(C(T, Z), pw)$ is angelical.*

Appendix D

Mixed topologies on vector spaces.

This appendix is based on the document [30] by A. Wiweger. In the whole appendix we will denote by F a vector space on $\mathbb{K}$ non-trivial. If τ is a vector topology in F (not necessarily Hausdorff), then, we will denote by $\mathcal{U}(\tau)$ a base of balanced neighborhoods of 0 for τ, such that for each $U \in \mathcal{U}(\tau)$ and $\lambda \in \mathbb{K}$, $\lambda \neq 0$, we have that $\lambda U \in \mathcal{U}(\tau)$. Also, if Z is a subset of F, then, the topology in Z induced by τ will be denoted by $\tau|Z$. The collection of all the τ-bounded subsets of (F, τ) will be denoted by $Bd(\tau)$.

Let τ and $\tau*$ be two vector topologies in F. For each sequence $U_n^* \in \mathcal{U}(\tau*)$ and for each $U \in \mathcal{U}(\tau)$, we will denote by $\gamma(U_1^*, U_2^*, \ldots; U)$ or simply by U^γ the set

$$\bigcup_{n=1}^{\infty}(U_1^* \cap U + U_2^* \cap 2U + \ldots + U_n^* \cap nU), \tag{D.1}$$

This is, the set of all the sums $x_1 + x_2 + \ldots + x_n$ $(n = 1, 2, \ldots)$, where $x_k \in U_k^*$ and $\frac{1}{k}x_k \in U$.

We will denote by $\mathscr{R}$ the family of all the sets of the form (D.1).

Lemma D.1. *The family $\mathscr{R}$ is a base of neighborhoods of* 0 *for a vector topology in F.*

Proof. It is enough to prove the following statements:

(a) For each $U^\gamma \in \mathscr{R}$ and $\lambda \in \mathbb{K}$, $\lambda \neq 0$, we have that $\lambda U^\gamma \in \mathscr{R}$.

(b) Each $U^\gamma \in \mathscr{R}$ is a balanced and absorbent set.

(c) Given $U^\gamma, V^\gamma \in \mathscr{R}$, there exists $W^\gamma \in \mathscr{R}$ such that $W^\gamma \subset U^\gamma \cap V^\gamma$.

Let $U^\gamma \in \mathscr{R}$ and $\lambda \in \mathbb{K}$, $\lambda \neq 0$. We have that:

$$\begin{aligned}\lambda U^\gamma = \lambda \cdot \gamma(U_1^*, U_2^*, \ldots; U) &= \lambda \cdot \bigcup_{n=1}^{\infty} (U_1^* \cap U + U_2^* \cap 2U + \ldots + U_n^* \cap nU) \\ &= \bigcup_{n=1}^{\infty} (\lambda U_1^* \cap \lambda U + \lambda U_2^* \cap 2\lambda U + \ldots + \lambda U_n^* \cap n\lambda U) \\ &= \gamma(\lambda U_1^*, \lambda U_2^*, \ldots; \lambda U).\end{aligned}$$

Thereby we have proved (a). From the previous equality, it follows that U^γ is balanced since the sets U_k^* ($k \in \mathbb{N}$) and U are balanced. Also, we have that U^γ is absorbent, in fact, given $x \in F$, there exists $\lambda \in \mathbb{K}$ such that $\lambda x \in U_1^* \cap U \subset U^\gamma$. Therefore (b) is satisfied. To prove (c), we consider $\gamma(U_1^*, U_2^*, \ldots; U), \gamma(V_1^*, V_2^*, \ldots; V) \in \mathscr{R}$. We choose $W \in \mathcal{U}(\tau)$, such that $W \subset U \cap V$, and for each $k \in \mathbb{N}$, we choose $W_k^* \in \mathcal{U}(\tau^*)$ such that $W_k^* \subset U_k^* \cap V_k^*$.

Then, $\gamma(W_1^*, W_2^*, \ldots; W) \subset \gamma(U_1^*, U_2^*, \ldots; U) \cap \gamma(V_1^*, V_2^*, \ldots; V)$. □

Definition D.2. *The vector topology in F that have $\mathscr{R}$ as base of neighborhoods of* 0*, is called **mixed topology** determined by the vector topologies τ and τ^* and denoted by $\gamma\,[\tau, \tau^*]$.*

Lemma D.3. *We have that $\tau^* \leqslant \gamma\,[\tau, \tau^*]$.*

Proof. Given $U^* \in \mathcal{U}(\tau^*)$, there exists $U_1^* \in \mathcal{U}(\tau^*)$ such that $U_1^* + U_1^* \subset U^*$. Also, there exists $U_2^* \in \mathcal{U}(\tau^*)$ such that $U_2^* + U_2^* \subset U_1^*$. Recursively, for each $n \in \mathbb{N}$, there exists $U_n^* \in \mathcal{U}(\tau^*)$ such that $U_n^* + U_n^* \subset U_{n-1}^*$. Thereby, we have that $U_1^* + U_2^* + \ldots + U_n^* \subset U^*$ for each $n \in \mathbb{N}$ and therefore, $\gamma(U_1^*, U_2^*, \ldots; U) \subset U^*$ for each $U \in \mathcal{U}(\tau)$. □

Lemma D.4. *If $\tau^* \leqslant \tau$, then, $\gamma\,[\tau, \tau^*] \leqslant \tau$.*

Proof. Given $\gamma(U_1^*, U_2^*, \ldots; U) \in \mathscr{R}$, there exists $V \in \mathcal{U}(\tau)$ such that $V \subset U_1^* \cap U \subset \gamma(U_1^*, U_2^*, \ldots; U)$. □

Let τ, τ^* and τ' be three vector topologies in F. We will say that the topology τ' satisfies the condition (P_1) (with respect to the pair (τ, τ^*)) if: $\tau'|Z = \tau^*|Z$ for each $Z \in Bd(\tau)$.

Lemma D.5. *The mixed topology $\gamma[\tau, \tau^*]$ satisfies the condition (P_1).*

Proof. Let $Z \in Bd(\tau)$. By the lemma D.3, we have that $\tau^*|Z \leqslant \gamma[\tau, \tau^*]|Z$.

To prove the other inequality, consider $x \in Z$ and $\gamma(U_1^*, U_2^*, \ldots; U) \in \mathscr{R}$. Then $(x + \gamma(U_1^*, U_2^*, \ldots; U)) \cap Z$ is a neighborhood of x respect to the topology $\gamma[\tau, \tau^*]|Z$. Since $-x + Z \in Bd(\tau)$, there exists $n \in \mathbb{N}$ such that $-x + Z \subset nU$. Then, if $z \in (x + U_n^*) \cap Z$, then, we can write $z = x + y$, where $y \in U_n^*$. Since $y = -x + z \in -x + Z \subset nU$, it is concluded that $y \in U_n^* \cap nU$. Thereby, $(x + U_n^*) \cap Z \subset (x + U_n^* \cap nU) \cap Z \subset (x + \gamma(U_1^*, U_2^*, \ldots; U)) \cap Z$. Therefore, $\gamma[\tau, \tau^*]|Z \leqslant \tau^*|Z$. $\square$

Lemma D.6. *If the vector topology τ is such that $\mathcal{U}(\tau) \subset Bd(\tau)$, then, for each vector topology τ' defined in F, the condition*

$$\tau'|Z \leqslant \tau^*|Z \quad \textit{for each} \quad Z \in Bd(\tau)$$

implies the relation

$$\tau' \leqslant \gamma[\tau, \tau^*]$$

Proof. Let τ' be a vector topology in F such that $\tau'|Z \leqslant \tau^*|Z$ for each $Z \in Bd(\tau)$. Let $V' \in \mathcal{U}(\tau')$ and $U \in \mathcal{U}(\tau)$. Since $U \in Bd(\tau)$, we have that $\bigoplus_{i=1}^{n} U \in Bd(\tau)$ for each $n \in \mathbb{N}$. Due to the condition that satisfies τ', for each $n \in \mathbb{N}$, there exists $U_n^* \in \mathcal{U}(\tau^*)$ such that

$$U_n^* \cap \bigoplus_{i=1}^{\frac{n(n+1)}{2}} U \subset V' \cap \bigoplus_{i=1}^{\frac{n(n+1)}{2}} U \ .$$

By the same argument that was used in D.3, we can choose a sequence $W_n^* \in \mathcal{U}(\tau^*)$

such that $W_1^* + W_2^* + \ldots + W_n^* \subset U_n^*$, for each $n \in \mathbb{N}$. Then, for $n \in \mathbb{N}$, we have

$$\begin{aligned} W_1^* \cap U + W_2^* \cap 2U + \ldots + W_n^* \cap nU &\subset (W_1^* + W_2^* + \ldots + W_n^*) \cap \bigoplus_{i=1}^{\frac{n(n+1)}{2}} U \\ &\subset U_n^* \cap \bigoplus_{i=1}^{\frac{n(n+1)}{2}} U \\ &\subset V' \cap \bigoplus_{i=1}^{\frac{n(n+1)}{2}} U \end{aligned}$$

Thereby,

$$\bigcup_{n=1}^{\infty}(W_1^* \cap U + W_2^* \cap 2U + \ldots + W_n^* \cap nU) \subset \bigcup_{n=1}^{\infty} \left(V' \cap \bigoplus_{i=1}^{\frac{n(n+1)}{2}} U \right) \subset V' \ .$$

Therefore, V' is a neighborhood of 0 for $\gamma\,[\tau, \tau^*]$ and therefore, $\tau' \leqslant \gamma\,[\tau, \tau^*]$. □

Corollary D.7. *If the vector topology τ is such that $\mathcal{U}(\tau) \subset Bd(\tau)$, then, the mixed topology $\gamma\,[\tau, \tau^*]$, is the finest of all the vector topologies that satisfy the condition (P_1).*

Proof. Is immediate from the two previous lemmas. □

For each sequence $U_n^* \in \mathcal{U}(\tau^*)$ $(n = 0, 1, 2, \ldots)$ and for each $U \in \mathcal{U}(\tau)$, we will denote by $\gamma_1(U_0^*, U_1^*, \ldots; U)$ or simply by U^{γ_1} to the set

$$U_0^* \cap \bigcap_{n=1}^{\infty}(nU + U_n^*) \ . \tag{D.2}$$

The family of all sets of the form (D.2) will be denoted by $\mathscr{R}_1$.

Additionally, if $(a_n)_{n\in\mathbb{N}}$ is a a sequence of positive real numbers tending to infinity, then we will denote by U^{γ_2} the set

$$U_0^* \cap \bigcap_{n=1}^{\infty}(a_n U + U_n^*) \ . \tag{D.3}$$

The family of all sets of the form (D.3) will be denoted by $\mathscr{R}_2$.

Lemma D.8. *Given $U^{\gamma_1} \in \mathscr{R}_1$, there exists $U^{\gamma_2} \in \mathscr{R}_2$ such that $U^{\gamma_2} \subset U^{\gamma_1}$. Conversely, if $U^{\gamma_2} \in \mathscr{R}_2$, there exists $U^{\gamma_1} \in \mathscr{R}_1$ such that $U^{\gamma_1} \subset U^{\gamma_2}$.*

Proof. Let $U^{\gamma_1} = U_0^* \cap \bigcap_{n=1}^{\infty}(nU + U_n^*)$. Consider a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers tending to infinity. There exists a sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ strictly increasing, such that $a_1 = k_1$ and $a_n \leqslant k_n$ for each $n \in \mathbb{N}$.

We choose $V_0^*, V_1^*, V_2^* \ldots$ in $\mathcal{U}(\tau^*)$, such that

$$V_0^* \subset \bigcap_{p=0}^{k_1-1} U_p^* \qquad \text{and} \qquad V_n^* \subset \bigcap_{p=k_n}^{k_{n+1}-1} U_p^*\,, \quad n \in \mathbb{N}.$$

We have that: $V_0^* \subset U_0^*$ and $a_n U \subset pU$, for each $p \geqslant a_n$. Given a fix $n \in \mathbb{N}$, for each $p \in \mathbb{N}$ such that $k_n \leqslant p \leqslant k_{n+1} - 1$, we have: $a_n U + V_n^* \subset pU + U_p^*$. Therefore,

$$a_n U + V_n^* \subset \bigcap_{p=k_n}^{k_{n+1}-1} \left(pU + U_p^*\right) .$$

Thus,

$$\bigcap_{n=1}^{\infty} (a_n U + V_n^*) \subset \bigcap_{p=k_1}^{\infty} \left(pU + U_p^*\right) ,$$

then,

$$V_0^* \cap \bigcap_{n=1}^{\infty} (a_n U + V_n^*) \subset U_0^* \cap \bigcap_{p=k_1}^{\infty} \left(pU + U_p^*\right) ,$$

If $k_1 = 1$, then we have concluded that

$$U^{\gamma_2} := V_0^* \cap \bigcap_{n=1}^{\infty} (a_n U + V_n^*) \subset U_0^* \cap \bigcap_{p=k_1}^{\infty} \left(pU + U_p^*\right) = U^{\gamma_1} .$$

If $k_1 > 1$, then $\quad V_0^* \subset \bigcap_{p=0}^{k_1-1} U_p^* \subset \bigcap_{p=1}^{k_1-1} U_p^* \subset \bigcap_{p=1}^{k_1-1} \left(U_p^* + pU\right)$. Then,

$$U^{\gamma_1} := V_0^* \cap \bigcap_{n=1}^{\infty} (a_n U + V_n^*) \subset U_0^* \cap V_0^* \cap \bigcap_{p=k_1}^{\infty} \left(pU + U_p^*\right) \subset U_0^* \cap \bigcap_{p=1}^{\infty} \left(pU + U_p^*\right) = U^{\gamma_2} .$$

For the converse, let $U^{\gamma_2} = U_0^* \cap \bigcap_{n=1}^{\infty}(a_n U + U_n^*)$, where $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers tending to infinity. There exists a sequence of positive integers

$(N_n)_{n\in\mathbb{N}}$ strictly increasing, such that for each $n \in \mathbb{N}$, we have that: $m \geqslant N_n \Rightarrow n \leqslant a_m$. For each $n \in \mathbb{N}$, we choose $k_n \in \mathbb{N}$, $k_n \geqslant N_n$, such that the sequences $(k_n)_{n\in\mathbb{N}}$ and $(a_{k_n})_{n\in\mathbb{N}}$ are strictly increasing and $k_1 > 1$. Note that, for each $n \in \mathbb{N}$, we have: $m \geqslant k_n \Rightarrow n \leqslant a_m$. We choose $V_0^*, V_1^*, V_2^* \ldots$ in $\mathcal{U}(\tau^*)$, such that

$$V_0^* \subset \bigcap_{p=0}^{k_1-1} U_p^* \qquad \text{and} \qquad V_n^* \subset \bigcap_{p=k_n}^{k_{n+1}-1} U_p^*\,, \quad n \in \mathbb{N}.$$

We have that: $V_0^* \subset U_0^*$ and $nU \subset a_pU$, for each $p \geqslant k_n$. Given a fix $n \in \mathbb{N}$, for each $p \in \mathbb{N}$ such that $k_n \leqslant p \leqslant k_{n+1}-1$, we have that: $nU + v_n^* \subset a_pU + U_p^*$. Therefore,

$$nU + V_n^* \subset \bigcap_{p=k_n}^{k_{n+1}-1} \left(a_pU + U_p^*\right)\,.$$

Thus,

$$\bigcap_{n=1}^{\infty} (nU + V_n^*) \subset \bigcap_{p=k_1}^{\infty} \left(a_pU + U_p^*\right)\,,$$

then,

$$V_0^* \cap \bigcap_{n=1}^{\infty} (nU + V_n^*) \subset U_0^* \cap V_0^* \bigcap_{p=k_1}^{\infty} \left(a_pU + U_p^*\right)\,,$$

Since, $V_0^* \subset \bigcap_{p=0}^{k_1-1} U_p^* \subset \bigcap_{p=1}^{k_1-1} U_p^* \subset \bigcap_{p=1}^{k_1-1} \left(U_p^* + a_pU\right)$, it is concluded that

$$U^{\gamma_1} := V_0^* \cap \bigcap_{n=1}^{\infty} (nU + V_n^*) \subset U_0^* \cap \bigcap_{p=1}^{\infty} \left(a_pU + U_p^*\right) = U^{\gamma_2}\,.$$

□

Proposition D.9. *Suppose that the topology τ is locally convex and all the members of $\mathcal{U}(\tau)$ with convex sets. The families $\mathscr{R}_1$ and R_2 are basis of neighborhood of 0 for the mixed topology $\gamma\,[\tau,\tau^*]$.*

Proof. Due to the previous lemma, it is enough to prove that the family of all the sets of the form

$$U_0^* \cap \bigcap_{n=1}^{\infty}(\frac{1}{2}n(n+1)U + U_n^*) \ . \tag{D.4}$$

is a base of neighborhoods of 0 for the topology $\gamma\,[\tau,\tau^*]$. First, we prove that each set of the form (D.4) is a neighborhood of 0 for $\gamma\,[\tau,\tau^*]$.

Let $V_1^* \in \mathcal{U}(\tau^*)$ be such that $V_1^* + V_1^* \subset U_0^*$. Taking recursively $V_n^* \in \mathcal{U}(\tau^*)$ $(n > 1)$ such that $V_n^* + V_n^* \subset U_{n-1} \cap V_{n-1}^*$.

Then we have that

$$V_1^* + V_2^* + \ldots + V_n^* \subset U_0^* \tag{D.5}$$

and for $p \in \mathbb{N}$, we have that

$$V_n^* + V_{n+1}^* + \ldots + V_{n+p}^* \subset V_n^* + V_n^* \subset U_{n-1}^* \tag{D.6}$$

By (D.6), for each $n > 1$, we obtain

$$\begin{aligned}\gamma(V_1^*, V_2^*, \ldots; U) = \bigcup_{p=1}^{\infty}(V_1^* \cap U + V_2^* \cap 2U + \ldots + V_{n-1}^* \cap (n-1)U + V_n^* \cap nU + \ldots \\ \ldots + V_{n+p}^* \cap (n+p)U) \\ \subset \bigcup_{p=1}^{\infty}(U + 2U + \ldots (n-1)U + V_n^* + \ldots + V_{n+p}^*) \\ \subset \frac{1}{2}n(n-1)U + U_{n-1}^* \ .\end{aligned}$$

From the above, together with (D.5), it follows that

$$\gamma(V_1^*, V_2^*, \ldots; U) \subset U_0^* \cap \bigcap_{n=1}^{\infty}\left(\frac{1}{2}n(n+1)U + U_{n+1}^*\right) \ .$$

Now we will prove that each set $U^\gamma \in \mathscr{R}$ contains a set of the form (D.4). Let $U^\gamma = \gamma(U_1^*, U_2^*, \ldots; U)$ be a neighborhood of 0 for the topology $\gamma\,[\tau,\tau^*]$. For $n \in \mathbb{N}$, we write $m_n = 2n - 1$. We can choose a sequence $V_i^* \in \mathcal{U}(\tau^*)$ $(i = 0, 1, 2 \ldots)$ such that $V_{n-1}^* + V_{n-1}^* \subset U_{m_n}^*$ y $V_n^* \subset V_{n-1}^*$, $n \in \mathbb{N}$.

We will prove that $U^{\gamma_1} := \gamma(V_0^*, V_1^*, \ldots; U) \subset U^\gamma$. Let $x \in U^{\gamma_1}$. Then $x \in V_0^*$ and for each $n \in \mathbb{N}$, there exists a decomposition $x = y_n + z_n$, where $y_n \in nU$ and $z_n \in V_n^*$.

Let $x_1 = y_1$ and $x_n = y_n - y_{n-1}$, for $n > 1$. For each $n > 1$, we have the identity:

$$x_1 + x_2 + \ldots + x_n + z_n = y_1 + (y_2 - y_1) + \ldots + (y_n - y_{n-1}) + z_n = y_n + z_n = x \quad \text{(D.7)}$$

Also, we have that $X = zn + y_n = z_{n-1} + y_{n-1}$. Then, $x_n = y_n - y_{n-1} = z_{n-1} - Z_n \in V^*_{n-1} + V^*_n$. On the other hand, $x_n = y_n - y_{n-1} \in nU + (n-1)U = (2n-1)U = m_n U$. From this, it follows that $x_n \in (V^*_{n-1} + V^*_n) \cap m_n U$. Since, $V^*_{n-1} + V^*_n \subset V^*_{n-1} + V^*_{n-1} \subset U^*_{m_n}$. Thus, for $n > 1$, we have that

$$x_n \in U^*_{m_n} \cap m_n U\,. \quad \text{(D.8)}$$

From the equality $z_n = x - y_n$, we have that $z_n \in (k_0 + n)U$, where $k_0 \in \mathbb{N}$ such that $x \in k_0 U$. If $n_0 > k_0 - 1$, then $2n_0 + 1 = m_{n_0} > k_0 + n_0$ and $z_n \in V^*_{n_0} \subset V^*_{n_0} + V^*_{n_0} \subset U^*_{m_{n_0+1}}$. Therefore,

$$z_{n_0} \in U^*_{m_{n_0+1}} \cap m_{n_0+1} U\,. \quad \text{(D.9)}$$

By (D.7), (D.8) and (D.9), we have that

$$x = x_1 + \ldots + x_{n_0} + z_{n_0} \in U^*_{m_1} \cap m_1 U + \ldots + U^*_{m_{n_0}} \cap m_{n_0} U + U^*_{m_{n_0+1}} \cap m_{n_0+1} U \subset U^\gamma.$$

Therefore, $U^{\gamma_1} \subset U^\gamma$, concluding the proof. □

Theorem D.10. *Suppose that a vector space F, the topology τ is generated by a seminorm $\|\cdot\|$, and the topology τ^* by a family of seminorms $\{\|\cdot\|_\alpha : \alpha \in \Lambda\}$.*

If the seminorms $\|\cdot\|$ and $\|\cdot\|_\alpha$ satisfy the conditions:

(a) $\|x\| = \sup_{\alpha\in\Lambda} \|x\|_\alpha$, for each $x \in F$, and

(b) If $\{\alpha_n : n \in \mathbb{N}\} \subset \Lambda$, $x \in F$ and $\varepsilon > 0$, then, for each $n \in \mathbb{N}$, there exist $y, z \in F$, such that $x = y + z$, $\|z\|_{\alpha_i} = 0$ for $i = 1, 2, \ldots, n$, and $\|y\| \leqslant \text{máx}\{\|x\|_{\alpha_1}, \ldots, \|x\|_{\alpha_n}\} + \varepsilon\,,$

then, the family of all the set of the form

$$\bigcap_{i=1}^{\infty} \{x \in F : \|x\|_{\alpha_i} \leqslant a_i\}, \quad \text{(D.10)}$$

where $0 < a_i \to \infty$, is a base of neighborhoods of 0 for the mixed topology $\gamma\,[\tau, \tau^*]$.

Proof. Using a similar reasoning to the one used in D.1, it follows that the family of all the sets of the form D.10, is a base of neighborhoods of 0 of a vector topology τ_1 en F. Let $Z \in Bd(\tau)$, this is, $Z \subset \{x \in F : \|x\| \leqslant r\}$, for some $r > 0$. Taking any neighborhood of an element $x_0 \in Z$ in the toplogy $\tau_1|Z$. We can assume that this neighborhood is of the form

$$Z \cap \bigcap_{i=1}^{\infty} \{x \in F : \|x - x_0\|_{\alpha_i} \leqslant a_i\}, \qquad 0 < a_i \to \infty, \qquad \alpha_i \in \Lambda\,.$$

If x, x_0, then, by virtue of (a), $\|x - x_0\|_{\alpha_i} \leqslant \|x - x_0\| \leqslant 2r$. From the condition $a_i \to \infty$, it follows that there exists $i_0 \in \mathbb{N}$ such that $a_i > 2r$, for $i \geqslant i_0$. Thereby,

$$Z \cap \bigcap_{i=1}^{i_0} \{x \in F : \|x - x_0\|_{\alpha_i} \leqslant a_i\} = Z \cap \bigcap_{i=1}^{\infty} \{x \in F : \|x - x_0\|_{\alpha_i} \leqslant a_i\}.$$

From this, it is concluded that $\tau_1|Z \leqslant \tau^*$. Then, by D.6, we have that $\tau_1 \leqslant \gamma\,[\tau, \tau^*]\,$.

By D.9, every neighborhood of 0 in the topology $\gamma\,[\tau, \tau^*]$, contains a set of the form

$$U_0^* \cap \bigcap_{n=1}^{\infty} (nU + U_n^*), \tag{D.11}$$

where $U = \{x \in F : \|x\| \leqslant r\}$, $r > 0$, $U_n^* = \{x \in F : \max\limits_{1 \leqslant i \leqslant k_n} \|x\|_{\alpha_i} \leqslant \varepsilon_n\}$, $\alpha_i \in \Lambda$, $\varepsilon_n > 0$, $k_n < k_{n+1}$ for $n = 0, 1, 2, \ldots$

Let $a_i = \min\{\varepsilon_0, r/2\}$ be for $1 \leqslant i \leqslant k_0$ and $a_i = \frac{1}{2}nr$ for $k_{n-1} \leqslant i \leqslant k_n$.

Let $x \in \bigcap_{i=1}^{\infty} \{x \in F : \|x\|_{\alpha_i} \leqslant a_i\}$. Since $\|x\|_{\alpha_i} \leqslant a_i \leqslant \varepsilon_0$ for $1 \leqslant i \leqslant k_0$, we have that $x \in U_0^*$. Let $m \in \mathbb{N}$. By the condition (b), there exist elements $y, z \in F$, such that $x = y + z$, $\|z\|_{\alpha_i} = 0$ for $i = 1, 2, \ldots, k_m$, and $\|y\| \leqslant \max\limits_{1 \leqslant i \leqslant k_m} \{\|x\|_{\alpha_i}\} + \varepsilon\,$. Then, $z \in U_m^*$ and $\|y\| \leqslant \frac{1}{2}mr + \frac{1}{2}mr = mr$, this is $y \in mU$.

Thereby, $x \in U_m^* + mU$. By the arbitrariness of $m \in \mathbb{N}$, it follows that $x \in U_0^* \cap \bigcap_{n=1}^{\infty} (nU + U_n^*)$. Therefore

$$\bigcap_{i=1}^{\infty} \{x \in F : \|x\|_{\alpha_i} \leqslant a_i\} \subset U_0^* \cap \bigcap_{n=1}^{\infty} (nU + U_n^*),$$

and from this, it is concluded that $\gamma\,[\tau, \tau^*] \leqslant \tau_1$. □

Appendix E

Open Questions.

During the development of this thesis, there were some questions that were not answered in the covered topics and neither are mentioned the responses in the used literature.

(a) Under what conditions the space $(C_b(X,E), \beta_{\mathcal{P}})$ is Mackey ?.

(b) In section 2.7 it is shown that the topologies β_p and $\gamma\,[u_p, \tau_p]$ coincide in the space $C_b(X,E)$.

  (i) Does the topology $\gamma\,[\tau_u, \tau_{\mathcal{P}}]$ coincide with the topology $\beta_{\mathcal{P}}$?.

  (ii) Is the topology $\beta_{\mathcal{P}}$ the finest vector topology which coincide with the topology $\tau_{\mathcal{P}}$ in the τ_u-bounded sets ?.

(c) How are the extensions of the measures of the space $\mathcal{M}_{\mathcal{P},t}(X, \mathscr{L}(E,F))$ to σ-algebra generated by $\mathcal{Z}$? Which properties satisfy the integral operators induced by these new measures ?.

Index of Notations

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