UNIVERSIDAD DE CONCEPCIÓN
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
PROGRAMA DE DOCTORADO EN MATEMÁTICA

TEORÍA ULTRAMÉTRICA DE DISTRIBUCIÓN DE VALORES EN VARIAS VARIABLES Y ALGUNAS CONSECUENCIAS EN LÓGICA

(ULTRAMETRIC VALUE DISTRIBUTION THEORY IN SEVERAL VARIABLES AND SOME CONSEQUENCES IN LOGIC)

Tesis para optar al grado de
Doctor en Matemática

JOSÉ LUIS RIQUELME BECERRA
CONCEPCIÓN-CHILE
2015

Profesor Guía: Xavier Vidaux
Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas
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Teoría ultramétrica de distribución de valores en varias variables y algunas consecuencias en definibilidad

Ultrametric value distribution theory in several variables and some consequences in definability

José Luis Riquelme Becerra

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Introducción

En este trabajo, estudiamos algunas propiedades aritméticas de los anillos de funciones enteras y sus consecuencias en definibilidad y decidibilidad en lógica. Antes de poder enunciar nuestros resultados, es necesario introducir algunas notaciones y un poco de vocabulario.

Sea $\mathbb{F}$ un campo ultramétrico completo el cual es algebraicamente cerrado. Sea $p \geq 0$ la característica de $\mathbb{F}$. Denotamos por $\mathcal{E}_n$ el anillo de las funciones enteras en las n variables $x_1, \dots, x_n$ sobre $\mathbb{F}$. El campo de fracciones de $\mathcal{E}_n$ será denotado por $\mathcal{M}_n$ y lo llamaremos el campo de las funciones meromorfas en varias variables. Recordamos que cada función meromorfa no nula, puede ser representada como un cociente f_0/f_∞ de funciones enteras coprimas. Módulo multiplicación por una constante, las funciones enteras f_0 y f_∞ están determinadas de manera única por f. Decimos que f_0 es la *función de ceros de* f y que f_∞ es la *función de polos de* f. El conjunto de las funciones enteras irreducibles que dividen a f será denotado por $\mathrm{supp}(f)$ y lo llamaremos el *soporte de zeros de* f, o simplemente el *soporte de* f. Cuando $\mathbb{F}$ es de característica p positiva, el *índice de ramificación de* f de una función meromorfa no constante f es el mayor entero m tal que $f = g^{p^m}$ para algún $g \in \mathcal{M}_n$. El índice de ramificación de f será denotado por $u(f)$. Cuando $p = 0$, escribimos $u(f) = 0$ para cualquier función meromorfa no constante. Finalmente, cuando $n = 1$, recordamos que dos funciones meromorfas f y g *comparten un valor* b cuando $f^{-1}(b) = g^{-1}(b)$.

Un teorema muy conocido de R. Nevanlinna (por ejemplo, ver [Hay1, Theorem 2.6]) establece que dos funciones meromorfas no constantes, en una variable compleja, las cuales comparten cinco valores distintos, deben ser idénticas. El teorem análogo para funciones meromorfas globales sobre un campo no-Arquimediano (completo y algebraicamente cerrado) de característica cero fue demostrado por W. W. Adams y E. G. Straus [AdSt]. El caso de característica positiva fue resuelto por A. Boutabaa and A. Escassut [BoEs] (con una hipótesis adicional). Más precisamente, en esta tesis llamaremos Teorema de Adams-Straus-Boutabaa-Escassut (Teorema AS-BE) al siguiente enunciado:

> Sean f y g dos funciones meromorfas sobre un campo ultramétrico completo, el cual es algebraicamente cerrado. Adicionalmente, cuando $\mathbb{F}$ es de característica positiva, asumimos que f y g tienen el mismo índice de ram-

ificación. Si f y g comparten cuatro valores distintos, entonces f y g son idénticas. Si más aun, f y g son funciones enteras, entonces es suficiente que f y g compartan dos valores para asegurar que son idénticas.

El Teorema AS-BE es generalizado para funciones enteras en la sección 3.4, y luego para funciones meromorfas en la sección 4.3. [1]

Teorema 1. *Sean f y g funciones enteras no constantes sobre $\mathbb{F}^n$ tales que $u(f) = u(g)$. Si existen dos valores distintos $a, b \in \mathbb{F}$ tales que*

$$\mathrm{supp}(f - a) = \mathrm{supp}(g - a) \qquad \textit{and} \qquad \mathrm{supp}(f - b) = \mathrm{supp}(g - b),$$

entonces $f = g$.

Una pregunta natural que surge en característica positiva es cuan diferentes pueden ser f y g si en el Teorema 1, la condición $u(f) = u(g)$ no se satisface. El siguiente corolario da una respuesta a esta pregunta.

Corolario 2. *Asumimos que $\mathbb{F}$ es de característica positiva. Sean f y g dos funciones enteras no constantes sobre $\mathbb{F}^n$, las cuales satisfacen $u(f) \geq u(g)$. Sea $M = u(f) - u(g)$. Si existen dos valores distintos a y b en el subcampo finito $\mathbb{F}_{p^{u(g)}}$ de $\mathbb{F}$, los cuales satisfacen*

$$\mathrm{supp}(f - a) = \mathrm{supp}(g - a) \qquad \textit{and} \qquad \mathrm{supp}(f - b) = \mathrm{supp}(g - b),$$

entonces $f = g^{p^M}$.

Notamos que cuando $u(f) = u(g)$, el resultado anterior es simplemente el Teorema 1.

Ahora, enunciamos los resultados análogos para funciones meromorfas.

Teorema 3. *Sean f y g dos funciones enteras no constantes sobre $\mathbb{F}^n$, las cuales satisfacen $u(f) = u(g)$. Si existen cuatro valores $a_1, a_2, a_3, a_4 \in \mathbb{F}$ tales que*

$$\mathrm{supp}(f - a_i) = \mathrm{supp}(g - a_i)$$

para cada $i \in \{1, 2, 3, 4\}$, entonces $f = g$.

La demostración del siguiente corolario es similar a la demostración del Corolario 2 (al final de la Sección 3.4), usando el Teorema 3 en lugar del Teorema 1.

[1]Agradecemos a Julie Tzu-Yueh Wang por señalarnos que el caso de varias variables puede obtenerse del caso de una variable. De cualquier manera, nuestro enfoque proporciona una demostración *uniforme*, en el sentido que la demostración misma no depende del número de variables, y por tanto, puede considerarse como una demostración alternativa.

Corolario 4. *Asumimos que $\mathbb{F}$ es de característica positiva. Sean f y g dos funciones meromorfas no constantes tales que $u(f) \geq u(g)$. Sea $M = u(f) - u(g)$. Adicionalmente, cuando p es dos o tres, asumimos $u(g) > 1$. Si existen cuatro valores distintos a_1, a_2, a_3, a_4 en el campo finito $\mathbb{F}_{p^{u(g)}}$ que satisfacen*

$$\operatorname{supp}(f - a_i) = \operatorname{supp}(g - a_i)$$

para cada $i = 1, 2, 3, 4$, entonces $f = g^{p^M}$.

Nuestra principal motivación detrás de este resultado viene de una conjetura clásica de la teoría de números, conocida como la Conjetura de Erdös-Woods, la cual (informalmente) establece lo siguiente:

> Existe un entero positivo N tal que cada entero k está completamente determinado por la lista de los divisores primos de k y sus N sucesores $x + 1, \ldots, x + N$.

El siguiente corolario puede pensarse como un análogo de la Conjetura de Erdös-Woods sobre el anillo de las funciones enteras ultramétricas (ver [Vs, Theorem 2] para un análogo polinomial, el cual inspiró parte de esta tesis).

Teorema 5. *Dadas dos funciones enteras f y g, si existen dos constantes $a, b \in \mathbb{F}$ y un polinomio de grado uno $\ell \in \mathbb{F}[x_1, \ldots, x_n]$ tal que*

- $\operatorname{supp}(f) = \operatorname{supp}(g)$,
- $\operatorname{supp}(f + a) = \operatorname{supp}(g + a)$,
- $\operatorname{supp}(f + \ell) = \operatorname{supp}(g + \ell)$ *and*
- $\operatorname{supp}(f + \ell + b) = \operatorname{supp}(g + \ell + b)$,

entonces $f = g$.

El resultado anterior tiene implicaciones en aspectos de la *lógica* de las funciones enteras en cualquier característica. En la sección 4.3 se demuestra una versión meromorfa del Teorema 5 (ver Corolario 4.3.4), con consecuencias análogas en lógica. No obstante, tales consecuencias pueden ser obtenidas directamente del Teorema 3, por lo que las omitiremos.

En este trabajo, usaremos $\mathscr{L}$ para denotar el lenguaje de primer orden $\{S, S_*, \perp\}$, donde S y S_* son dos símbolos de función unaria y $\perp$ es un símbolo de relación binaria (para una breve introdcucción a las nociones básicas de la lógica de predicados de primer orden, ver Sección 3.1). Denotamos por Ω la clase de anillos de funciones enteras en un número arbitrario de variables, sobre un campo ultramétrico arbitrario completo y algebraicamente cerrado. Por anillo de funciones enteras, siempre nos referimos a un anillo de la clase Ω. Cada anillo de funciones enteras será considerado como una estructura sobre $\mathscr{L}$, donde $\perp$ es interpretado como la relación de coprimalidad, y los símbolos de funciones S y S_* son interpretados como las operaciones $S(x) = x + 1$ y $S_*(x) = x + X$, respectivamente, donde X denota cualquier variable independiente del anillo.

Como consequencia del Teorema 5, deducimos el siguiente corolario..

Corolario 6. *La relación de igualdad es uniformemente $\mathscr{L}$-definible en la clase Ω.*

Inspirados por el caso polinomial, resuelto por M. Vsemirnov en [Vs], demostramos el siguiente resultado.

Teorema 7. *Sea $\mathcal{E}$ cualquier anillo de funciones enteras de característica positiva. La teoría elemental de $\langle \mathcal{E}; \perp, S, S_* \rangle$ es indecidible.*

Estos resultados dan respuestas a preguntas análogas a tópicos clásicos estudiados, en particular, por Julia Robinson y Alain Woods para el caso de números enteros - Ver Sección 3.2 para más detalles.

Regresamos ahora a algunos tópicos del análisis no-Arquimediano, los cuales están relacionados con la Teoría de Distribución de Valores. Este es el contenido del capítulo 4. A pesar de los avances realizados en el análisis no-Arquimediano en varias variables, algunas herramientas necesarias para probar algunos resultados de esta tesis, o bien no habían sido desarrolladas o no eran lo suficientemente convenientes (especialmente para la teoría de los *valores ramificados* y sus aplicaciones). En lugar de seguir el enfoque propuesto por W. Cherry y Z. Ye [ChYe] (ver tambien [Kh]), donde se definen las funciones de conteo de ceros y polos a travez de una *recta general*, optamos por proponer un enfoque más al estilo Grothendieck definiendo la noción de *punto irreducible* (el cual es una clase de equivalencia de funciones enteras irreducibles). Dicha noción nos permitió obtener una definición bastante natural del concepto general de divisor asociado a una función entera en varias variables. La ventaja de este enfoque es la facilidad con que nos permite deducir las fórmulas clásicas de la teoría de la Nevanlinna al estilo de Boutabaa y Escassut [BoEs] (las cuales son válidas en el caso de una variable y en cualquier característica). En la Sección 4.2, reproducimos la teoría ultramétrica de Nevanlinna. Nada es realmente nuevo en esta sección, la cual incluimos (con demostraciones detalladas) con el proposito de ser auto-contenidos y porque no pudimos hallar en la literatura los resultados escritos de esta forma (al estilo de Boutabaa y Escassut) y con tal generalidad (en cualquier característica)[2]. Estos resultados son precisamente los que necesitamos y que usaremos en las secciones que siguen y cuyo contenido describimos a continuación.

[2]Julie Tzu-Yueh Wang nos ha mencionado un artículo de Cherry y Toropu [ChTo]. Demuestran que el análogo de la n-conjetura – una versión generalizada de la conjetura ABC propuesta por Browkin y Brzezinski – para funciones enteras ultramétricas, es cierta. Para $n = 3$, su teorema es equivalente al corolario de nuestro Segundo Teorema Principal truncado con 3 targets – ver Corolario 4.2.9

Antes de proceder, es necesario introducir algunas definiciones y notaciones. Recordamos que en el caso de una variable, $\mathcal{M}(\mathbb{F})$ denota el campo de las funciones meromorfas sobre $\mathbb{F}$ y $\mathcal{M}(d(a, r^-))$ denota el campo de las funciones meromorfas en el disco sin borde $d(a, r^-)$, con centro en a y radio r. Notaciones análogas son usadas para funciones análiticas usando la letra $\mathcal{A}$ en lugar de $\mathcal{M}$. Añadimos a estas notaciones un índice b para referirnos a funciones acotadas, y u para referirnos a las no-acotadas.

En [EsOj], Alain Escassut y Jacqueline Ojeda consideran la noción de *valor perfectamente ramificado* para funciones en $\mathcal{M}(\mathbb{F})$ o en $\mathcal{M}(d(a, r^-))$ (ver [Cha] para el caso complejo). Esta noción está estrechamente relacionada con el valor excepcional de Picard [Es3, Hay2]. A continuación recordamos dicha noción.

> Dada una función meromorfa sobre $\mathbb{F}$ o en $d(a, r^-)$, decimos que $b \in \mathbb{F}$ es un *valor perfectamente ramificado* para f cuando el conjunto de los ceros simples de $f - b$ es finito. Si $f - b$ no tiene ceros simples, decimos que b es un *valor perfectamente ramificado* para f.

Extendemos esta definición para funciones meromorfas globales en varias variables. Recordamos que la noción de *parte libre de cuadrado* de una función entera está bien definida (es el producto de los factores irreducibles cuya multiplicidad es uno).

Definición 8. Dada una función meromorfa no constante $f \in \mathcal{M}_n$, diremos que $b \in \mathbb{F}$ es un *valor perfectamente ramificado* para f si la parte libre de cuadrados de la función de ceros de $f - b$ es un polinomio. También, diremos que b es un *valor totalmente ramificado* para f si la parte libre de cuadrados de $f - b$ es una constante.

Con esta definición, extendemos algunos de los resultados obtenidos en [EsOj] para funciones meromorfas globales en varias variables y en cualquier característica. En la Sección 4.4, demostramos el siguiente teorema.

Teorema 9. *Dada una función meromorfa no constante f la cual satisface $u(f) = 0$, se tiene:*

1. *f tiene a lo más tres valores perfectamente ramificadoss;*
2. *si f es trascendental, entonces tiene a lo más cuatro valores perfectamente ramificados;*
3. *si f es trascendental y su función de polos es un polinomio, entonces tiene a lo más un valor perfectamente ramificado;*
4. *si f es un polinomio, entonces tiene a lo más un valor totalmente ramificado.*

En [EsRi] junto con Alain Escasssut, usamos el análogo del Teorema 9 en el caso de una variable y característica cero (previamente obtenido en [EsOj]) para analizar ciertas ecuaciones funcionales. Algunos de los resultados obtenidos en [EsRi], son presentados en esta tesis en el caso más general de varias variables y en cualquier característica - ver Sección 4.5. Recordamos que el *exponente característico* de $\mathbb{F}$ se define como p cuando p es positivo, y como 1 cuando p es cero. A lo largo de esta tesis, el exponente característico será denotado por q.

Corolario 10. *Sean m y d dos enteros mayores a 1, ambos coprimos con q, y sean $P, Q, R \in \mathbb{F}[x_1, \ldots, x_n]$ tres polinomios no nulos. La ecuación funcional*

$$Pf^d - Qg^m = R$$

en las variables f y g, no admite soluciones trascendentales en $\mathcal{E}_n$.

Teorema 11. *Sea $m \geq 2$ un entero coprimo con q. Sea $h \in \mathcal{E}_n$ una potencia m-ésima y $Q \in \mathbb{F}[X_1, \ldots, X_n]$, ambos no nulos. La ecuación funcional*

$$g^m = hf^m + Q$$

en las variables f y g no admite soluciones trascendentales sobre $\mathcal{E}_n$. Más aun, si h es trascendental, entonces no admite soluciones.

El resto de los resultados obtenidos en [EsRi] tratan con funciones analíticas en una variable, sobre discos sin borde. La razón por la cual no pudimos generalizar estos resultados a varias variables fue porque no pudimos resolver el problema de Lazard [La] en este contexto[3]. Las demostraciones de los siguientes teoremas están dadas en la Sección 4.6. Antes de establecer dichos resultados, recordamos que la *característica residual* del campo ultramétrico $\mathbb{F}$, es la característica de su campo residual.

Teorema 12. *Asumimos que $\mathbb{F}$ tiene característica cero. Fijamos $a \in \mathbb{F}$ y $\rho \in \mathbb{R}_{>0}$. Sean $m \in \mathbb{Z}_{\geq 2}$. Sean $h \in \mathcal{A}(d(a, \rho^-))$ y $w \in \mathcal{A}_b(d(a, \rho^-))$ ambos no nulos. La ecuación funcional*

$$g^m = hf^m + w$$

en las variables f y g no admite soluciones en $\mathcal{A}_u(d(a, \rho^-))$ en cualquiera de los siguientes casos:

- *Si m es coprimo con la característica resdual de $\mathbb{F}$ y h no tiene ceros en $d(a, \rho^-)$;*

[3] Recordamos que en el caso de una variable (con $\mathbb{F}$ esfericamente completo), Lazard [La] demostró que dado un divisor de un disco sin borde (esto es, una sucesión de elementos del disco, con multiplicidades, cuya sucesión de valores absolutos tiende al radio del disco), existe una función analítica en ese disco cuyos ceros – contando multiplicidades – son los preescritos por el divisor. Hasta donde sabemos, este problema es abierto en el caso de varias variables. Nuestra definición de divisor, dada en la Sección 4.1, nos permite resolver este problema de una manera bastante sencilla – ver Teorema 4.1.4.

- *si m y la característica residual de $\mathbb{F}$ son iguales a 2, y para cada $x \in d(a, \rho^-)$ se tiene*
$$|h(x) - h(0)| < \frac{1}{2}|h(0)|;$$
- *si m es coprimo con la característica resdual de $\mathbb{F}$ y cada cero de h tiene orden de multiplicidad divisible por m. Más aun, si también asumimos que h es no acotada, entonces la ecuación funcional no tiene solución.*

Teorema 13. *Asumimos que $\mathbb{F}$ tiene característica cero. Fijamos $a \in \mathbb{F}$ y $\rho \in \mathbb{R}_{>0}$. Sean $n, m \in \mathbb{Z}_{\geq 2}$ tales que $\max\{n, m\} \geq 3$. Dados $h, w \in \mathcal{A}_b(d(a, \rho^-))$, la ecuación funcional*
$$g^n = hf^m + w$$
en las variables f y g no admite soluciones en $\mathcal{A}_u(d(a, \rho^-))$.

La versión en una variable del Corolario 10 se obtuvo como un intento para hallar las soluciones f y g en $\mathcal{A}(\mathbb{F})$ de la ecuación funcional de Pell
$$f^2 - (h^2 - 1)g^2 = 1, \tag{1}$$
donde h es una función entera no constante arbitraria, pero fija. Nuestra motivación general viene del hecho que si las soluciones son lo *suficientemente buenas*, entonces se puede obtener un resultado de indecidibilidad bastante fuerte. Si por ejemplo, se puede probar que todas las soluciones son polinomios en h, entonces la teoría positivo existencial de $\mathcal{A}(\mathbb{F})$, en el lenguaje $\mathcal{L}_T$ de anillos junto con un predicado T para "x es una función no constante", sería indecidible. L. Rubel formuló un pregunta que necesita este lenguaje para poder formularse. En [Ru], él demostró que un anillo de funciones analíticas complejas en el disco unitario, abierto o cerrado, tiene teoría existencial decidible sobre $\mathcal{L}_T$. Para una discusión sobre este lenguaje, ver [PhZa3].

Cuando h es un polinomio no constante, el Corolario 10 reduce el problema de resolver la ecuación (2) al caso polinomial, el cual fue resuelto por T. Pheidas y K. Zahidi en [PhZa, Lem 2.2]. Se sabe bastante de este problema cuando $h = z$ es la variable independiente del anillo $\mathcal{A}(\mathbb{F})$. Primero, J. Denef usó las soluciones polinomiales de la ecuación (2) para obtener resultados de indecidibilidad para anillos de polinomios (en cualquier característica). Sobre un anillo de funciones enteras ultramétricas, todas las soluciones son polinomiales (este es un resultado de L. Lipshitz y T. Pheidas [LiPh] en el caso de característica cero – también ver [Vi] para una demostración del resultado de indecidibilidad, usando curvas elípticas en lugar de ecuaciones de Pell –, y N. García-Fritz y H. Pasten [GaPa] cuando la característica es positiva y distinta de dos – en característica dos, ellos resuelven una ecuación análoga que ajusta mejor sus propositos para obtener los resultados de indecidibildad en los que están interesados). Cuando h es trascendental, aun no se sabe si las soluciones son polinomios en h. Para surveys más generales sobre el décimo problema de Hilbert, ver [PhZa2], [Po] y [Koe], y el libro [Sh].

En la Sección 4.5, también demostramos el siguiente corolario del Teorema 9 (inspirados por un resultado de N. García-Fritz - ver [Ga]).

Corolario 14. *Sean f y g funciones enteras coprimas tales que $\nabla\left(\frac{f}{g}\right) \neq 0$. Existen a lo más tres valores de λ en $\mathbb{F}$ tales que la función entera*

$$f + \lambda g$$

es una potencia en $\mathcal{E}_n$ (esto es que puede ser escrita como h^k para algun $h \in \mathcal{E}_n$ y algún $k \in \mathbb{Z}_{\geq 2}$).

Introduction

In this work, we study some arithmetical properties of rings of ultrametric entire functions and their consequences in definability and decision problems in mathematical logic. Before we state our results, we need to introduce some basic vocabulary and notation.

Let $\mathbb{F}$ be a complete ultrametric field which is algebraically closed. Let $p \geq 0$ be the characteristic of $\mathbb{F}$. We denote by $\mathcal{E}_n$ the ring of entire functions in the n variables x_1, $\ldots$, x_n over $\mathbb{F}$. The fraction field of $\mathcal{E}_n$ will be denoted by $\mathcal{M}_n$ and referred to as the field of meromorphic functions in several variables. Recall that each non-zero meromorphic function f can be represented as a quotient f_0/f_∞ of coprime entire functions. Up to multiplication by a constant, the entire functions f_0 and f_∞ are uniquely determined by f. We say that f_0 is the *function of zeros of* f, and that f_∞ is the *function of poles of* f. The set of irreducible entire functions dividing f_0 is denoted by $\mathrm{supp}(f)$ and is called the *support of zeros of* f, or simply *the support* of f. When the characteristic p is positive, the *ramification index* of a non-constant meromorphic function f is the largest integer m such that $f = g^{p^m}$ for some $g \in \mathcal{M}_n$. The ramification index of f will be denoted by $u(f)$. When $p = 0$, we set $u(f) = 0$ for any non-constant meromorphic function f. Finally, when $n = 1$, we recall that two meromorphic functions f and g *share a value* b if $f^{-1}(b) = g^{-1}(b)$.

A well known theorem of R. Nevanlinna (see for example [Hay1, Theorem 2.6]) states that two non-constant meromorphic functions of a complex variable which share five distinct values must be identical. The analogous theorem for global meromorphic functions over a non-Archimedean (complete and algebraically closed) field of characteristic zero was proven by W. W. Adams and E. G. Straus [AdSt]. The positive characteristic case was solved by A. Boutabaa and A. Escassut [BoEs] (with a necessary additional hypothesis). More precisely, in this thesis, what will be referred to as the Adams-Straus-Boutabaa-Escassut Theorem (AS-BE Theorem) is the following:

> Let f and g be two non constant meromorphic functions over a complete ultrametric field $\mathbb{F}$ which is algebraically closed. Additionally, if $\mathbb{F}$ has positive characteristic, assume that f and g have the same ramification index. If f and g share four distinct values, then f and g are identical. If moreover, f and g are entire functions, then sharing two values is enough to ensure

that they are identical.

We will generalize the AS-BE Theorem to the case of several variables, first for entire functions in Section 3.4, and then for meromorphic functions in Section 4.3.[4]

Theorem 1. *Let f and g be two non-constant entire functions over $\mathbb{F}^n$ such that $u(f) = u(g)$. If there are two distinct values $a, b \in \mathbb{F}$ such that*

$$\operatorname{supp}(f-a) = \operatorname{supp}(g-a) \qquad \textit{and} \qquad \operatorname{supp}(f-b) = \operatorname{supp}(g-b),$$

then $f = g$.

A natural question in the positive characteristic case, is how much different can be f and g if the condition $u(f) = u(g)$ is not satisfied in Theorem 1. The following corollary gives an answer to this question.

Corollary 2. *Assume that $\mathbb{F}$ has positive characteristic. Let f and g be two non-constant entire functions over $\mathbb{F}^n$ satisfying $u(f) \geq u(g)$. Write $M = u(f) - u(g)$. If there are two distinct values a and b in the finite subfield $\mathbb{F}_{p^{u(g)}}$ of $\mathbb{F}$ which satisfy*

$$\operatorname{supp}(f-a) = \operatorname{supp}(g-a) \qquad \textit{and} \qquad \operatorname{supp}(f-b) = \operatorname{supp}(g-b),$$

then $f = g^{p^M}$.

Note that when $u(f) = u(g)$, the previous statement is just Theorem 1.
Now we state the analogous statement for meromorphic functions.

Theorem 3. *Let f and g be two non-constant meromorphic functions over $\mathbb{F}^n$ such that $u(f) = u(g)$. If there are four distinct values $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that*

$$\operatorname{supp}(f-a_i) = \operatorname{supp}(g-a_i)$$

for each $i \in \{1, 2, 3, 4\}$, then $f = g$.

The proof of the following corollary is similar to the proof of Corollary 2 (at the end of Section 3.4), using Theorem 3 instead of Theorem 1.

Corollary 4. *Assume that $\mathbb{F}$ has positive characteristic. Let f and g be two non-constant meromorphic functions satisfying $u(f) \geq u(g)$. Write $M = u(f) - u(g)$. Additionally, when p is two or three, suppose $u(g) > 1$. If there are four distinct values a_1, a_2, a_3, a_4 in the finite field $\mathbb{F}_{p^{u(g)}}$ which satisfy*

$$\operatorname{supp}(f-a_i) = \operatorname{supp}(g-a_i)$$

for each $i = 1, 2, 3, 4$, then $f = g^{p^M}$.

[4]We thank Julie Wang for pointing out to us that the several variables case can be deduced from the one variable case. We had not realized it. Nevertheless, our approach gives a *uniform* proof, in the sense that the proof itself is independent of the number of variables, and therefore may be seen as an alternative proof.

Our main motivation behind these results comes from a classical number-theoretic conjecture, known as the Erdös-Woods Conjecture, which (informally) states:

> There is a positive integer N such that every integer k is completely determined by the list of prime divisors of k and its N successors $x+1,\dots,x+N$.

The following corollary can be thought of as an analogue of the Erdös-Woods Conjecture over the ring of ultrametric entire functions (see [Vs, Theorem 2] for a polynomial analogue, which inspired part of this thesis).

Theorem 5. *Given two entire functions f and g, if there are two constants $a, b \in \mathbb{F}$ and a one degree polynomial $\ell \in \mathbb{F}[x_1, \dots, x_n]$ such that*

- $\mathrm{supp}(f) = \mathrm{supp}(g)$,
- $\mathrm{supp}(f + a) = \mathrm{supp}(g + a)$,
- $\mathrm{supp}(f + \ell) = \mathrm{supp}(g + \ell)$ *and*
- $\mathrm{supp}(f + \ell + b) = \mathrm{supp}(g + \ell + b)$,

then $f = g$.

The previous result has implications in *logical* aspects of entire functions in any characteristic. A meromorphic version of Theorem 5 is given in Section 4.3 (see Corollary 4.3.4), with analogous consequences in logic. Nevertheless, those can be obtained directly from Theorem 3, so we will omit them.

In this work, we will use $\mathscr{L}$ to denote the first-order language $\{S, S_*, \perp\}$, where S and S_* are two unary function symbol and $\perp$ is a binary relation symbol (see Section 3.1 for a short introduction to the basic concepts of first-order predicate calculus that we need in this thesis). We will denote by Ω the class of rings of entire functions in an arbitrary number of variables, over an arbitrary complete ultrametric field which is algebraically closed. By a ring of entire functions, we will always mean a ring in the class Ω. Every ring of entire functions will be considered as a structure over $\mathscr{L}$, where $\perp$ is interpreted as the coprimeness relation, and the function symbols S and S_* are interpreted as the operations $S(x) = x + 1$ and $S_*(x) = x + X$, respectively, where X denotes an independent variable of the ring.

As a consequence of Theorem 5, we deduce the following corollary.

Corollary 6. *Equality is uniformly $\mathscr{L}$-definable in the class Ω.*

Inspired by the polynomial case, which is due to M. Vsemirnov in [Vs], we also prove the following results.

Theorem 7. *Let $\mathcal{E}$ be any ring of entire functions of positive characteristic. The elementary theory of $\langle \mathcal{E}; \perp, S, S_* \rangle$ is undecidable.*

These results answer questions that are analogous to classical topics studied, in particular, by Julia Robinson and Alan Woods in the case of integers - see Section 3.2 for more details.

We now come back to some other topics in non-Archimedean analysis which are related to Value Distribution Theory. This is the content of Chapter 4. Though non-Archimedean analysis in several variables has long been studied, some tools that we needed in order to prove some of the theorems in this thesis, were either missing or not convenient (especially for the theory of branched values and the applications that we wanted from it). Instead of following the approach by W. Cherry and Z. Ye [ChYe] (see also [Kh]), which for example define counting functions of zeros and poles through a general line, we opted for a more "Grothendieck like" approach, first defining a concept of irreducible point (which will be classes of equivalence of irreducible functions), which allows us to have a very natural definition of the general concept of divisor in several variables. The advantage of this approach is that we get very easily the "Boutabaa-Escassut like" classical formulas [BoEs] (which so far hold in the one variable case and in any characteristic). In Section 4.2, we reproduce the main theory of Nevanlinna. Nothing is really new in this section, which we include (with proofs and with full details) for the sake of being self-content, and because we could not find in the literature the statements written in this *form* (Boutabaa-Escassut like) and in such a generality.[5] These statements are precisely the ones that we need and that we will use for the sections that follow and whose content we now describe.

Before we proceed, we need to introduce a few more definitions and notation. Recall that in the one variable case, $\mathcal{M}(\mathbb{F})$ denotes the field of meromorphic functions on $\mathbb{F}$ and $\mathcal{M}(d(a, r^-))$ denotes the field of meromorphic functions in the stripped disc $d(a, r^-)$ with center $a \in \mathbb{F}$ and radius r. Analogous notation is used for analytic functions with the letter $\mathcal{A}$ instead of $\mathcal{M}$. We may add to the notation an index b to refer to bounded functions, and u to refer to unbounded functions.

In [EsOj], Alain Escassut and Jacqueline Ojeda consider the non-Archimedean analogue of the notion of *perfectly branched value* on $\mathcal{M}(\mathbb{F})$ and on $\mathcal{M}(d(a, r^-))$ (see [Cha] for the complex case). This notion is closely linked to Picard's exceptional values [Es3, Hay2]. Let us recall it here.

> Given a meromorphic function on $\mathbb{F}$ or in $d(a, r^-)$, a value $b \in \mathbb{F}$ is said to be a *perfectly branched value for f* when the set of simple zeros of $f - b$ is finite. In the case that $f - b$ has no simple zeros, b is said to be a *totally branched value* for f.

We extend these definitions to global meromorphic functions in several variables.

[5] Julie Wang has brought to our attention the paper by Cherry and Toropu [ChTo]. They prove that the analogue of the n-conjecture – a generalized version of the ABC conjecture proposed by Browkin and Brzezinski – for ultrametric entire functions, is true. For $n = 3$ their theorem is equivalent to the corollary of our truncated Second Main Theorem with 3 targets – see Corollary 4.2.9

Recall that the notion of the *square-free part* of an entire function is well defined (it is the product of the irreducible divisors of the function which have multiplicity one).

Definition 8. Given a non-constant meromorphic function $f \in \mathcal{M}_n$, we will say that $b \in \mathbb{F}$ is a *perfectly branched value for* f if the square-free part of the function of zeros of $f - b$ is a polynomial. Also, we will say that b is a *totally branched value for* f if the square-free part of the function of zeros of $f - b$ is a constant.

With this definition, we extend some of the results obtained in [EsOj] for goblal meromorphic functions to the several variables case and to any characteristic. In Section 4.4, we will prove the following theorem.

Theorem 9. *Given a non-constant meromorphic function f satisfying $u(f) = 0$, we have:*

1. *f has at most three totally branched values;*

2. *if f is transcendental, then it has at most four perfectly branched values;*

3. *if f is transcendental and its function of poles is a polynomial, then it has at most one perfectly branched value;*

4. *if f is a polynomial, then it has at most one totally branched value.*

In [EsRi], together with Alain Escassut, we use the one variable characteristic zero analogue of Theorem 9 (previously obtained in [EsOj]) to examine certain functional equations. Some of the results obtained in [EsRi] are presented in this thesis in the more general situation of several variables and any characteristic - see Section 4.5. Recall that the *characteristic exponent* of $\mathbb{F}$ is the integer defined as p if p is positive, and as 1 if p is zero. The characteristic exponent of $\mathbb{F}$ will be denoted by q throughout the thesis.

Corollary 10. *Let m and d be two integers greater than 1, both coprime with q, and let $P, Q, R \in \mathbb{F}[x_1, \ldots, x_n]$ be non-zero polynomials. The functional equation*

$$Pf^d - Qg^m = R$$

in the variables f and g has no transcendental solution over $\mathcal{E}_n$.

Theorem 11. *Let $m \geq 2$ be an integer coprime with q. Let $h \in \mathcal{E}_n$ be an m-th power and $Q \in \mathbb{F}[X_1, \ldots, X_n]$, both non-zero. The functional equation*

$$g^m = hf^m + Q$$

in the variables f and g has no transcendental solution over $\mathcal{E}_n$. Moreover, if h is transcendental, then it has no solution at all.

The other results obtained in [EsRi] deal with one variable analytic functions in stripped disks. The reason why we could not generalize these results to several variables is that we could not solve the analogue of Lazard's problem [La] in this context.[6] The proofs of the following theorems are given in Section 4.6. Before we state them, we recall that the *residual characteristic* of the ultrametric field $\mathbb{F}$ is the characteristic of its residual field.

Theorem 12. *Assume that $\mathbb{F}$ has characteristic zero. Fix $a \in \mathbb{F}$ and $\rho \in \mathbb{R}_{>0}$. Let $m \in \mathbb{Z}_{\geq 2}$. Let $h \in \mathcal{A}(d(a, \rho^-))$ and $w \in \mathcal{A}_b(d(a, \rho^-))$ be both non identically zero. The functional equation*

$$g^m = hf^m + w$$

in the variables f and g has no solution over $\mathcal{A}_u(d(a, \rho^-))$ in each of the following situation:

- *if m is coprime with the residual characteristic of $\mathbb{F}$ and h has no zeros in $d(a, \rho^-)$;*
- *if m and the residual characteristic of $\mathbb{F}$ are equal to 2, and for each $x \in d(a, \rho^-)$ we have*
$$|h(x) - h(0)| < \frac{1}{2}|h(0)|;$$
- *if m is coprime with the residual characteristic of $\mathbb{F}$ and each zero of h has order of multiplicity divisible by m. Moreover, if h is also assumed to be unbounded, then the functional equation has no solution.*

Theorem 13. *Assume that $\mathbb{F}$ has characteristic zero. Fix $a \in \mathbb{F}$ and $\rho \in \mathbb{R}_{>0}$. Let $n, m \in \mathbb{Z}_{\geq 2}$ be such that $\max\{n, m\} \geq 3$. Given $h, w \in \mathcal{A}_b(d(a, \rho^-))$, the functional equation*

$$g^n = hf^m + w$$

in the variables f and g has no solution over $\mathcal{A}_u(d(a, \rho^-))$.

The one variable version of Corollary 10 was obtained as an attempt to find the solutions f and g in $\mathcal{A}(\mathbb{F})$ of the functional Pell equation

$$f^2 - (h^2 - 1)g^2 = 1, \tag{2}$$

where h is a fixed arbitrary non-constant entire function. The general motivation comes from the fact that if the solutions turn to be *nice enough*, then one could obtain a very

[6] Recall that in the one variable case (with $\mathbb{F}$ spherically complete), Lazard [La] proved that, given a divisor of a stripped disk (a sequence of elements of the disk, with multiplicities, whose sequence of absolute values tends to the radius of the disk), there exists an analytic function on that disk whose zeros – counting multiplicities – are the ones prescribed by the divisor. As far as we know, this problem is open in the case of several variables. Our definition of divisors in Section 4.1 allows us to solve this problem for entire functions on $\mathbb{F}^n$ in a quite straightforward way – see Theorem 4.1.4.

strong undecidability result. If, for instance, one could prove that all the solutions are polynomials in h, then the positive existential theory of $\mathcal{A}(\mathbb{F})$, in the language $\mathcal{L}_T$ of rings together with a predicate T for "x is not a constant function", would be undecidable. L. Rubel asked a question which needs this language in order to be expressed. In [Ru], he showed that a ring of complex analytic functions on the open or closed unit disk has decidable existential theory over $\mathcal{L}_T$. See [PhZa3] for a discussion about this language.

When h is a non-constant polynomial, Corollary 10 reduces the problem of solving Equation (2) to the polynomial case, which was solved by T. Pheidas and K. Zahidi in [PhZa, Lem 2.2]. When $h = z$ is the independent variable of the ring $\mathcal{A}(\mathbb{F})$, much is known. First J. Denef [De] uses the polynomial solutions to Equation (2) to obtain undecidability results for polynomial rings (in any characteristic). Over a ring of ultrametric entire functions, all solutions turn out to be polynomial (this is a result of L. Lipshitz and T. Pheidas [LiPh] in the characteristic zero case — see also [Vi] for an alternative proof of the undecidability result, using elliptic curves instead of Pell equation —, and N. García-Fritz and H. Pasten [GaPa] when the characteristic is distinct from two - in characteristic two, they solve an analogous equation that fit better their purpose, as they are mostly interested in logical consequences). When h is transcendental, it is not known whether or not all solutions are polynomials in h. For general surveys on Hilbert's tenth problem, see [PhZa2], [Po] and [Koe], and the book [Sh].

In Section 4.5, we also prove the following corollary of Theorem 9 (inspired by a result due to N. García-Fritz - see [Ga]).

Corollary 14. *Let f and g be two coprime entire functions such that $\nabla\left(\frac{f}{g}\right) \neq 0$. There exist at most three values of λ in $\mathbb{F}$ such that the entire function*

$$f + \lambda g$$

is a power in $\mathcal{E}_n$ (meaning that it can be written as h^k for some $h \in \mathcal{E}_n$ and $k \in \mathbb{Z}_{\geq 2}$).

Chapter 1

Preliminaries in Non-Archimedean analysis

1.1 Non-Archimedean rings

In this Chapter we recall some basic properties of non-Archimedean rings. For most definitions and results, we refer to [BGR]. By a ring, we always mean a commutative ring with unit. Given a ring R, we use R^* to denote the set $R \setminus \{0\}$ and $R^\times$ for the group of units of R. A *non-Archimedean absolute value* on a ring R (also known as *ultrametric absolute value*) is a function

$$| \ |\colon R \to \mathbb{R}_{\geq 0}$$

which satisfies the following properties for every $x, y \in R$:

1. $|x| = 0$ if and only if $x = 0$,

2. $|xy| = |x| \cdot |y|$,

3. $|x + y| \leq \max\{|x|, |y|\}$.

The third property is known as the *strong triangle inequality.* A *non-Archimedean ring* (or an *ultrametric ring*) is a ring endowed with a non-Archimedean absolute value. It follows directly from Conditions 1 and 2 that a non-Archimedean ring is an integral domain.

Throughout this section, R will denote a ring endowed with a non-Archimedean absolute value $| \ |$, and K will denote its fraction field. The absolute value of R extends uniquely to a non-Archimedean absolute value $| \ |$ on K in the usual way. In this case, we say that K is an ultrametric field.

1.1.1 Basic facts and examples

The proof of the following is standard (sometimes known as "all triangles are isosceles").

Lemma 1.1.1. *Let x and y be two elements of R. If $|x| > |y|$, then $|x+y| = |x|$.*

We will often consider R as a topological ring and K as a topological field with respect to the distance induced by $|\ |$.

One defines a valuation v associated to $|\ |$ in the usual way: fix $\pi \in]1, +\infty[$, and consider the real logarithmic function $\log_\pi$ of base π. The function $v\colon R^* \to \mathbb{R}$, defined as $v(x) = -\log_\pi |x|$ satisfies the following properties: for every $x, y \in R^*$, we have

a) $v(xy) = v(x) + v(y)$, and

b) $v(x+y) \geq \min\{v(x), v(y)\}$.

Reciprocally, a valuation $v\colon R^* \to \mathbb{R}$ induces a non-Archimedean absolute value $|\ |$ in R by setting $|x| = \pi^{-v(x)}$. The metric topology induced by this absolute value does not depend on the choice of π.

We recall that any non-Archimedean ring can be embedded, as a dense subring, in a complete non-Archimedean ring.

Examples 1.1.2. 1. If A is an integral domain, the function $|\ |\colon A \to \mathbb{R}_{\geq 0}$ given by

$$|x| = \begin{cases} 1 & \text{if } \quad x \neq 0, \\ 0 & \text{if } \quad x = 0, \end{cases}$$

defines the *trivial* non-Archimedean absolute value. Any integral domain A with the trivial absolute value is complete.

2. Let $\mathbb{Z}$ be the ring of integers and let p be a prime number. The p-adic valuation v_p is defined as follows. Given a non-zero integer x, if s denotes the largest natural number such that x factorizes as $p^s q$ for some $q \in \mathbb{Z}$, then we set

$$v_p(x) = s.$$

The p-adic absolute value $|\ |_p$ is defined as the non-Archimedean absolute value associated to v_p. The completion of $\mathbb{Z}$ with respect to the p-adic absolute value is the usual ring $\mathbb{Z}_p$ of p-adic integers. The field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value and it is the fraction field of $\mathbb{Z}_p$.

3. The following example is especially relevant in positive characteristic. Let A be an integral domain. Consider the ring $A[[t]]$ of formal power series in the variable t with coefficients in A. The *order* of a non-zero power series is defined as the valuation

$$\mathrm{ord}\colon A[[t]] \setminus \{0\} \to \mathbb{Z}$$

given by
$$\operatorname{ord}\left(\sum_{i\in\mathbb{N}} a_i t^i\right) = \min\{i\colon a_i \neq 0\}.$$
The non-Archimedean absolute value on $A[[t]]$ associated to this valuation is denoted by $|\ |_0$, and $(A[[t]], |\ |_0)$ is a complete non-Archimedean ring. Moreover, $A[[t]]$ is the completion of the ring of polynomials $A[t]$ with respect to the absolute value $|\ |_0$, namely, it contains $A[t]$ as a dense subset.

We finish this subsection by recalling the following characterization of Cauchy sequences in ultrametric rings.

Proposition 1.1.3. *[BGR, 1.1.7, Prop. 1] Any sequence $(a_k)_{k\in\mathbb{N}}$ of elements of R is a Cauchy sequence if and only if it satisfies*
$$\lim_{k\to\infty} |a_{k+1} - a_k| = 0.$$

1.1.2 Complete non-Archimedean rings

In this subsection, we will assume that R is complete.

Lemma 1.1.4. *[BGR, 1.1.8, Prop. 1] A series $\sum\limits_{k=0}^{\infty} a_k$ of elements of R is convergent if and only if*
$$\lim_{k\to\infty} |a_k| = 0.$$

We notice that the strong triangle inequality can be extended to convergent series. More precisely, if $\sum_{k=0}^{\infty} a_k$ is a convergent series of elements in R, then
$$\left|\sum_{k=0}^{\infty} a_k\right| \leq \max_k |a_k|.$$

We recall that given a sequence $(a_k)_{k\in\mathbb{N}}$ of non-zero elements of R, the infinite product
$$\prod_{k=0}^{\infty} a_k$$
is said to be convergent if the sequence
$$\left(\prod_{i=0}^{k} a_k\right)_{k\in\mathbb{N}}$$
of its partial products is convergent.

Lemma 1.1.5. *Consider a sequence* $(a_k)_k$ *of non-zero elements in* R*. The infinite product*

$$\prod_{k=1}^{\infty} a_k$$

converges to a non-zero element of R *if and only if it satisfies*

$$\lim_{k\to\infty} |a_k - 1| = 0. \tag{1.1}$$

Proof. In [ChYe], the analogous statement for rings of analytic functions is proven. The same argument works for the more general case that is considered here. We give it for the convenience of the reader.

It is easily seen that an infinite product which converges to a non-zero element satisfies Equation (1.1). For the other direction, we prove that the sequence of the partial products $F_m = \prod_{i=1}^{m} a_i$ is a Cauchy sequence, and then we deduce the convergence of this sequence by the completeness of R. By condition (1.1), there exists a positive integer N such that $|a_k - 1| < 1$, for each $k \geq N$. Hence, by Lemma 1.1.1 it follows that $|a_k| = 1$, for each $k \geq N$. Consequently, we obtain the following relations

$$|F_{k+1} - F_k| = |F_k||a_{k+1} - 1| = |F_{N-1}||a_{k+1} - 1|,$$

for each $k \geq N$. Thus, by Equation (1.1) and Proposition 1.1.3, we conclude that $(F_i)_i$ is a Cauchy sequence. □

1.1.3 Ultrametric field extensions

In this subsection, we will recall a few facts in the special case of non-Archimedean fields.

For the following results we refer to [Es1]. In this subsection, $\mathbb{K}$ denotes an ultrametric field. Given $a \in \mathbb{K}$ and $r \in \mathbb{R}_{>0}$, the *dressed disk* with center a and radius r is the set

$$d(a, r) = \{x \in \mathbb{K} \colon |x - a| \leq r\}.$$

Analogously, the *stripped disk* with center a and radius r is defined as

$$d(a, r^-) = \{x \in \mathbb{K} \colon |x - a| < r\}.$$

Recall that both the dressed and the stripped disks are clopen.

Lemma 1.1.6. *The set* $d(0, 1)$ *is a local ring whose maximal ideal is* $d(0, 1^-)$*. Consequently, the quotient*

$$\widetilde{\mathbb{K}} = d(0, 1)/d(0, 1^-)$$

is a field.

The field $\widetilde{\mathbb{K}}$ is called the *residual class field of* $\mathbb{K}$. If x is an element of $d(0,1)$, its reduction modulo $d(0,1^-)$ is denoted by $\tilde{x}$.

Lemma 1.1.7. *[Es1, Th 6.3] If $\mathbb{K}$ is algebraically closed then its group of values $|K^*|$ is dense in $\mathbb{R}_{>0}$.*

Proposition 1.1.8. *[Es1, Th 6.3] Assume that $\mathbb{K}$ is complete and let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. There is a unique ultrametric absolute value $|\ \ |'$ of $\overline{\mathbb{K}}$ which extends the one of $\mathbb{K}$. Moreover, if a is an element of $\overline{\mathbb{K}}$, P is the minimal polynomial of a over $\mathbb{K}$ and d is the degree of P, then*

$$|a|' = \sqrt[d]{|P(0)|}.$$

Lemma 1.1.9. *[Es1, Th 6.6] Assume that $\mathbb{K}$ is complete and let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. Then the residual field of $\overline{\mathbb{K}}$ is an algebraic closure of $\widetilde{\mathbb{K}}$.*

Theorem 1.1.10. *[Es1, Th 6.10] If $\mathbb{K}$ is algebraically closed, then the completion of $\mathbb{K}$ is algebraically closed.*

Examples 1.1.11.

1. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value $|\ \ |_p$ on $\mathbb{Q}_p$ can be extended uniquely to an ultrametric absolute value of $\overline{\mathbb{Q}}_p$. The completion of the ultrametric field $\overline{\mathbb{Q}}_p$ is denoted by $\mathbb{C}_p$, and is called the field of complex p-adic numbers. By theorem 1.1.10, the ultrametric field $\mathbb{C}_p$ is complete and algebraically closed. The residual field of $\mathbb{C}_p$ is an algebraic closure of the finite field $\mathbb{F}_p$.

2. Let K be $\mathbb{Q}$ or $\mathbb{F}_p$. For convenience we say that K has characteristic p by considering $p = 0$ in the case that $K = \mathbb{Q}$. Consider the field $K((t))$, i.e. the fraction field of the ring $K[[t]]$. The absolute value $|\ \ |_0$ extends uniquely to an algebraic closure

$$\overline{K((t))}$$

of $K[[t]]$. The completion of $\overline{K((t))}$ is denoted by $\mathbb{C}_{\infty,p}$ and its residual field is an algebraic closure of K, so it is a field of characteristic p.

An ultrametric field $(K, |\ \ |)$ is said to be *spherically complete* when all decreasing sequences of closed disk have a non-empty intersection. It is immediate that a spherically complete ultrametric field is complete.

Theorem 1.1.12. *[Es1, Th 7.4] Each ultrametric field K admits an ultrametric field extension $\widehat{K}$ which is algebraically closed and spherically complete, with non-countable residual field and group of values $\mathbb{R}_{>0}$.*

1.2 Rings of power series

Let n be a positive integer and consider the n-tuple of indeterminates

$$X = (X_1, \dots, X_n).$$

Given any field K, we will denote by $K[[X]]$ the ring of formal power series in the n indeterminates $X_1, \dots, X_n$ over K. It is convenient to introduce the multi-index notation.

A *multi-index* (of length n) is an element of the additive monoid $\mathbb{N}^n$. We will use θ to denote the neutral element. The *degree* of a multi-index

$$\gamma = (\gamma_1, \dots, \gamma_n)$$

is the non-negative integer $|\gamma|$, defined by

$$|\gamma| = \gamma_1 + \cdots + \gamma_n.$$

The degree can be seen as a monoid homomorphism from $(\mathbb{N}^n, +, \theta)$ to $(\mathbb{N}, +, 0)$. It extends to a unique $\mathbb{Z}$-module homomorphism from $\mathbb{Z}^n$ to $\mathbb{Z}$. On $\mathbb{Z}^n$, we define the binary relation $\preceq$ as follows: if $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ are two elements of $\mathbb{Z}^n$, we write $\alpha \prec \beta$ if $\alpha \neq \beta$ and

- either $|\alpha| < |\beta|$, or
- $|\alpha| = |\beta|$ and $\alpha_i < \beta_i$, where i is the smallest subscript satisfying $\alpha_i \neq \beta_i$.

We write $\alpha \preceq \beta$ if $\alpha \prec \beta$ or $\alpha = \beta$. The relation $\preceq$ is a total order on $\mathbb{Z}^n$ called the *graded-lexicographical order*. Note that it is compatible with addition, and that it becomes a well-ordering on the set $\mathbb{N}^n$.

For any $\gamma \in \mathbb{N}^n$, we use X^γ to denote the monomial

$$X_1^{\gamma_1} \dots X_n^{\gamma_n}.$$

Recall that the *order* of a non-zero power series

$$f = \sum_\gamma a_\gamma X^\gamma$$

in $K[[X]]$ is the integer defined by

$$\operatorname{ord}(f) = \min\{|\gamma| \colon a_\gamma \neq 0\},$$

i.e. the least degree of a multi-index whose coefficient in f is non-zero. It is not difficult to prove that given two non-zero elements f and g in $K[[X]]$, we have

- $\operatorname{ord}(f+g) \geq \min\{\operatorname{ord}(f), \operatorname{ord}(g)\}$,
- $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$.

For each $j = 1, \ldots, n$, the j-th formal partial derivative of a power series f will be denoted by $\partial_j f$. We will denote by ∇f the n-tuple

$$(\partial_1 f, \ldots, \partial_n f)$$

of power series.

1.2.1 Generalized topologically pure extensions

Let $(\mathbb{K}, |\ |)$ be a complete ultrametric field. Given a positive real number ρ, we denote by $T_{n,\rho}$ the $\mathbb{K}$-algebra

$$\left\{ \sum_{\gamma} a_\gamma X^\gamma \in \mathbb{K}[[X]] \colon \lim_{|\gamma| \to \infty} |a_\gamma| \rho^{|\gamma|} = 0 \right\}.$$

Such algebra is endowed with the *generalized Gauss norm*

$$|\ |_\rho \colon T_{n,\rho} \to \mathbb{R}_{\geq 0}$$

defined as

$$\left| \sum_{\gamma} a_\gamma X^\gamma \right|_\rho = \max_{\gamma} |a_\gamma| \rho^{|\gamma|}.$$

As in Lemmas 1.17 and 1.18 in [HuYa], we have:

Lemma 1.2.1. *The norm $|\ |_\rho$ is an ultrametric absolute value over $T_{n,\rho}$. The ultrametric ring $T_{n,\rho}$ is complete. Given a power series*

$$f = \sum_{\gamma} a_\gamma X^\gamma$$

in $T_{n,\rho}$, the sequence (f_i) of polynomials defined by

$$f_i = \sum_{|\gamma| \leq i} a_\gamma X^\gamma,$$

converges to f.

Recall that for $\rho = 1$, $T_{n,\rho}$ is a topologically pure extension of $\mathbb{K}$ of degree n and is denoted by T_n. If ρ is in the the group of values of $\mathbb{K}$, for any $b \in \mathbb{K}$ satisfying $|b| = \rho$, the mapping

$$\Psi_b \colon T_{n,\rho} \to T_n$$

defined by

$$\Psi_b\left(\sum_\gamma a_\gamma X^\gamma\right) = \sum_\gamma a_\gamma b^{|\gamma|} X^\gamma,$$

is a $\mathbb{K}$-algebra isomorphism and an isometry between

$$(T_{n,\rho}, |\ \ |_\rho) \quad \text{and} \quad (T_n, |\ \ |_1).$$

Proposition 1.2.2. *[BGR, 5.1.3, prop 1] Assume that $\rho \in |\mathbb{K}^*|$. A power series*

$$f = \sum_\gamma a_\gamma X^\gamma$$

in $T_{n,\rho}$ is a unit if and only if

$$a_\theta \neq 0 \qquad \textit{and} \qquad |f - a_\theta|_\rho < |a_\theta|.$$

We end this subsection with the fact that topologically pure extensions are unique factorization domains. This statement is proven by Paolo Salmon in [Sa].

Proposition 1.2.3. *[Es2, Th 33.7] Assume that $\rho \in |\mathbb{K}^*|$. The ring $T_{n,\rho}$ is a unique factorization domain.*

1.2.2 Analytic functions

Here, we recall the relation between $T_{n,\rho}$ and the ring of analytic functions in the poly-disk of radius ρ (see Proposition 1.2.4).

Let $(\mathbb{F}, |\ \ |)$ be a complete ultrametric field which is algebraically closed. The $\mathbb{F}$-vector space $\mathbb{F}^n$ is provided with the Gauss norm

$$\|(x_1, \ldots, x_n)\| = \max_{1 \leq i \leq n} |x_i|.$$

Given $\rho > 0$, we denote by D^n_ρ the dressed poly-disk

$$\{z \in \mathbb{F}^n \colon \|z\| \leq \rho\}.$$

Note that if ρ is not in the group of values $|\mathbb{F}^*|$, the poly-disk D^n_ρ coincide with the stripped poly-disk

$$D^n_{\rho^-} = \{z \in \mathbb{F}^n \colon \|z\| < \rho\}.$$

An *analytic function* in D^n_ρ is a map $h\colon D^n_\rho \to \mathbb{F}$ which admits a power series expansion (in n variables). The $\mathbb{F}$-algebra of analytic functions in D^n_ρ is denoted by $\mathcal{A}(D^n_\rho)$. The $\mathbb{F}$-algebra of bounded analytic functions in D^n_ρ is denoted by $\mathcal{A}_b(D^n_\rho)$. The supremum norm on $\mathcal{A}_b(D^n_\rho)$ will be denoted by $\|\ \ \|_{D^n_\rho}$. By Lemma 1.1.4, each power series $f = \sum_\gamma a_\gamma X^\gamma$ in $T_{n,\rho}$ defines by evaluation an analytic function in D^n_ρ. The following proposition is well known. For instance, see [An, Lemma 2.4] and for the one variable case, see [Es1, Th 13.1].

Proposition 1.2.4. *Let ρ be a positive number. Each power series $f \in T_{n,\rho}$ is a bounded analytic function in D^n_ρ, and for each $r \in\ ]0,\rho]$, it satisfies*

$$\|f\|_{D^n_r} = |f|_r.$$

If $\rho \in |\mathbb{F}^*|$, then

$$T_{n,\rho} = \mathcal{A}_b(D^n_\rho) = \mathcal{A}(D^n_\rho).$$

Chapter 2

Non-Archimedean analysis in several variables

2.1 Entire and meromorphic functions

In this section we introduce the basic definitions and properties of ultrametric entire and meromorphic functions in several variables. In what follows, $(\mathbb{F}, |\ \ |)$ will denote a complete ultrametric field which is algebraically closed and n will denote a positive integer. Classically, an *entire function* is defined as a map $h : \mathbb{F}^n \to \mathbb{F}$ whose restriction to any polydisk D_r^n is analytic. The ring of entire functions on $\mathbb{F}^n$ will be identified with the ring $\mathcal{E}_n(\mathbb{F}) \subset \mathbb{F}[[X]]$ given by

$$\mathcal{E}_n(\mathbb{F}) = \bigcap_{r \in \mathbb{R}_{>0}} T_{n,r}.$$

The fraction field of $\mathcal{E}_n(\mathbb{F})$ will be denoted by $\mathcal{M}_n(\mathbb{F})$ and is called the *field of meromorphic functions* in n variables over $\mathbb{F}$. When it is clear from the context what the field $\mathbb{F}$ is, the ring of entire functions is denoted by $\mathcal{E}_n$ and the field of meromorphic functions by $\mathcal{M}_n$.

Consider the family of norms

$$\mathscr{F} = \{|\ \ |_r : r \in \mathbb{R}_{>0}\}$$

on $\mathcal{E}_n$. The ring of entire functions is naturally endowed with the ultrametric locally convex topology τ induced by $\mathscr{F}$ (see [PeSc], 3.7.2). This topology provides a topological ring structure on $\mathcal{E}_n$. A sequence of entire functions converges in $(\mathcal{E}_n, \tau)$ if and only if it converges in $(T_{n,r}, |\ \ |_r)$, for each $r > 0$. As a direct application of Lemma 1.2.1, we have

Lemma 2.1.1. *The ring of entire functions is a complete topological ring. Given an entire function $f = \sum_\gamma a_\gamma X^\gamma$, the sequence of polynomials*

$$f_i = \sum_{|\gamma| \leq i} a_\gamma X^\gamma,$$

converges to f.

In other words, $\mathcal{E}_n$ is a completion of $\mathbb{F}[X_1, \ldots, X_n]$ with respect τ.

2.2 The maximum term

In this section, we recall the main properties of the maximum term of an entire function (also known as the maximum modulus).

To each non-zero entire function

$$f = \sum_\gamma a_\gamma X^\gamma,$$

we associate the real valued function

$$|f| \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

defined by

$$|f|(r) = |f|_r, \qquad \text{if } r > 0$$

and

$$|f|(0) = \left(\max_{|\gamma| = \mathrm{ord}(f)} |a_\gamma| \right) \lim_{s \to 0^+} s^{\mathrm{ord}(f)}$$

(there should be no ambiguity with the notation $|\ |$ used also for the absolute value over $\mathbb{F}$). The real function $|f|$ is called *the maximum term of f*. It is easily checked that the maximum term is continuous and non-decreasing. To each $r \in \mathbb{R}_{>0}$, we associate the set of multi-indices

$$\Gamma_r(f) = \{\gamma \colon |a_\gamma| r^{|\gamma|} = |f|_r\},$$

and the two multi-indices

$$\nu^+(r, f) = \max \Gamma_r(f)$$

and

$$\nu^-(r, f) = \min \Gamma_r(f),$$

where the maximum and the minimum are taken with respect to the graded lexicographical order on $\mathbb{N}^n$. This definition is extended to $\mathbb{R}_{\geq 0}$, by setting

$$\Gamma_0(f) = \left\{ \gamma \colon |a_\gamma| = \max_{|\beta| = \mathrm{ord}(f)} |a_\beta| \right\},$$

$$\nu^+(0,f) = \max \Gamma_0(f)$$

and

$$\nu^-(0,f) = \theta.$$

A *critical radius* of f is a non-negative real number r' such that

$$|\nu^-(r',f)| < |\nu^+(r',f)|.$$

Since $\mathbb{F}$ is algebraically closed, each critical radius is contained in the group of values $|\mathbb{F}^*|$. From Proposition 1.2.2, we deduce:

Lemma 2.2.1. *A non-zero entire function f is a unit in $T_{n,\rho}$ if and only if it has no critical radius in the interval $[0,\rho]$.*

The following propositions seem to be well-known but we were not able to find a reference. The proofs of these facts are given in the next subsection.

Proposition 2.2.2. *Let f be a non-zero entire function.*

1. *For each $\rho > 0$, the interval $[0,\rho]$ contains only finitely many critical radii of f.*
2. *If f has infinitely many critical radii, then the increasing sequence $(\rho_i)_i$ of critical radii tends to $+\infty$.*
3. *If $r', r'' \in \mathbb{R}_{\geq 0}$ are such that the interval $]r', r''[$ contains no critical radius of f, then for each $r \in]r', r''[$ we have*
   a) $|\nu^+(r',f)| = |\nu^-(r,f)| = |\nu^+(r,f)| = |\nu^-(r'',f)|$, *and*
   b) $\Gamma_r(f) = \left\{\gamma\colon |a_\gamma| = \max\limits_{|\beta|=d} |a_\beta|\right\}$, *where* $d = |\nu^+(r',f)|$.

Let f be a non-zero entire function and $I \subset \mathbb{R}_{\geq 0}$ be an interval containing no critical radius of f. The *central degree* of f in I is defined as the integer number $|\nu^+(r',f)|$, where r' is any number in I. A multi-index ν is a *central multi-index* of f in I, if there is $r' \in I$ such that $\nu \in \Gamma_{r'}(f)$.

Proposition 2.2.3. *Let*

$$f = \sum_\gamma a_\gamma X^\gamma \quad \textit{and} \quad g = \sum_\gamma b_\gamma X^\gamma$$

be two non-zero entire functions and write

$$fg = \sum c_\gamma X^\gamma.$$

For any $r \in \mathbb{R}_{\geq 0}$, we have

1. $\nu^+(r, fg) = \nu^+(r, f) + \nu^+(r, g)$ *and* $\nu^-(r, fg) = \nu^-(r, f) + \nu^-(r, g)$,

2. $|c_{\nu^+(r,fg)}| = |a_{\nu^+(r,f)}||b_{\nu^+(r,g)}|$ *and* $|c_{\nu^-(r,fg)}| = |a_{\nu^-(r,f)}||b_{\nu^-(r,g)}|$,

3. *If* $f = cg$ *for some constant* $c \in \mathbb{F}$, *then*

$$\frac{|g|(r)}{|b_{\nu^+(0,g)}|} = \frac{|f|(r)}{|a_{\nu^+(0,f)}|}.$$

2.3 Proofs of Propositions 2.2.2 and 2.2.3

First we need a lemma.

Lemma 2.3.1. *Let* f *be a non zero entire function. Given* $r, r' \in [0, \rho]$ *such that* $r < r'$, *if* $\nu \in \Gamma_r(f)$ *and* $\nu' \in \Gamma_{r'}(f)$, *then* $|\nu| \leq |\nu'|$. *In particular, this implies that*

$$|\nu^+(r, f)| \leq |\nu^-(r', f)|.$$

Proof. Write

$$f = \sum_\gamma a_\gamma X^\gamma.$$

Let ν' be any multi-index in $\Gamma_{r'}(f)$. Since the inequality

$$|a_\gamma| r'^{|\gamma|} \leq |a_{\nu'}| r'^{|\nu'|}$$

is satisfied by any multi-index γ, in particular, for each γ whose degree is greater than $|\nu'|$ and $a_\gamma \neq 0$, we have

$$\frac{|a_\gamma|}{|a_{\nu'}|} r^{|\gamma|-|\nu'|} < \frac{|a_\gamma|}{|a_{\nu'}|} r'^{|\gamma|-|\nu'|} \leq 1,$$

which implies that

$$|a_\gamma| r^{|\gamma|} < |a_{\nu'}| r^{|\nu'|}.$$

Thus, the maximum term $|f|_r$ is not reached for terms $a_\gamma X^\gamma$ whose degree $|\gamma|$ is greater than $|\nu'|$. If $r = 0$, it is clear that

$$\mathrm{ord}(f) \leq |\nu'|$$

for any $\nu' \in \Gamma_{r'}(f)$. □

The following lemma is crucial.

Lemma 2.3.2. *Let f be a non-zero entire function. If $r' > 0$, then there exists $\zeta \in \mathbb{R}_{\geq 0}$ such that $0 < \zeta < r'$, which satisfies, for each $r \in \,]\zeta, r'[$,*

$$|\nu^-(r, f)| = |\nu^+(r, f)| = |\nu^-(r', f)|.$$

If $r' \geq 0$, then there exists ξ such that $r' < \xi$, which satisfies, for each $r \in \,]r', \xi[$,

$$|\nu^-(r, f)| = |\nu^+(r, f)| = |\nu^+(r', f)|.$$

Proof. Consider the multi-indices

$$\nu^+ = \nu^+(r', f) \quad \text{and} \quad \nu^- = \nu^-(r', f).$$

First, we prove the second assertion. Assume that the set of multi-indices

$$\Gamma_{>|\nu^+|} = \{\gamma \colon |\gamma| > |\nu^+| \text{ and } a_\gamma \neq 0\}$$

is non-empty. For each $\beta \in \Gamma_{>|\nu^+|}$, define

$$\xi_\beta = \left| \frac{a_{\nu^+}}{a_\beta} \right|^{\frac{1}{|\beta| - |\nu^+|}}.$$

Since

$$|a_\beta| r'^{|\beta|} < |a_{\nu^+}| r'^{|\nu^+|},$$

we have

$$\xi_\beta^{|\beta| - |\nu^+|} = \frac{|a_{\nu^+}|}{|a_\beta|} > r'^{|\beta| - |\nu^+|},$$

hence $\xi_\beta > r'$. Let $r \in \,]r', \xi_\beta[$. Since $\xi_\beta > r$, we have

$$\frac{|a_{\nu^+}|}{|a_\beta|} = \xi_\beta^{|\beta| - |\nu^+|} > r^{|\beta| - |\nu^+|}.$$

Hence, for any $\beta \in \Gamma_{>|\nu^+|}$ and for any $r \in \,]r', \xi_\beta[$, we have

$$|a_\beta| r^{|\beta|} < |a_{\nu^+}| r^{|\nu^+|}.$$

Next, we prove the following fact.

Fact. There exists $\xi \in \mathbb{R}_{>0}$ such that, for any γ whose degree is greater than $|\nu^+|$, and for any $r \in \,]r', \xi[$, we have

$$|a_\gamma| r^{|\gamma|} < |a_{\nu^+}| r^{|\nu^+|}.$$

Indeed, fix an arbitrary ρ greater than r'. When $\Gamma_{>|\nu^+|}$ is empty, choose $\xi = \rho$. Otherwise, when $\Gamma_{>|\nu^+|}$ is finite, this is achieved by choosing

$$\xi = \min \left\{ \rho, \min_{\beta \in \Gamma_{>|\nu^+|}} \xi_\beta \right\}.$$

Assume that $\Gamma_{>|\nu^+|}$ is infinite. From

$$|a_\beta| \rho^{|\beta|} = \frac{|a_{\nu^+}| \rho^{|\beta|}}{\xi_\beta^{|\beta|-|\nu^+|}} = |a_{\nu^+}| \rho^{|\nu^+|} \left(\frac{\rho}{\xi_\beta} \right)^{|\beta|-|\nu^+|},$$

we obtain

$$\lim_{|\beta| \to \infty} \left(\frac{\rho}{\xi_\beta} \right)^{|\beta|-|\nu^+|} = 0,$$

and consequently, there is a multi-index $\beta_0 \in \Gamma_{>|\nu^+|}$ such that, for any $\beta \in \Gamma_{>|\nu^+|}$ satisfying $|\beta| > |\beta_0|$, we have $\xi_\beta > \rho$. So in this case we can choose

$$\xi = \min \left\{ \rho, \min_{|\nu^+| < |\beta| < |\beta_0|} \xi_\beta \right\}.$$

From the Fact above, we deduce that there is no term $a_\gamma X^\gamma$ of degree greater than $|\nu^+|$ which reaches the maximum term $|f|_r$ when $r \in \,]r', \xi[$. Therefore, for all $r \in \,]r', \xi[$, we get

$$|\nu^+| \leq |\nu^-(r, f)| \leq |\nu^+(r, f)| \leq |\nu^+|,$$

where the first inequality comes from Lemma 2.3.1, and the last one from what we just proved. So the second assertion is proved.

The first assertion is proven in a very similar and much easier way, by defining

$$\Gamma_{<|\nu^-|} = \{\gamma \colon |\gamma| < |\nu^-| \text{ and } a_\gamma \neq 0\},$$

which is a finite set of multi-indices. □

Proof of Proposition 2.2.2. First note that Assertion (2) is immediate from Assertion (1).

Assertion (1) is an immediate consequence of Lemma 2.3.2, since it implies that the set C of critical radii of f is discrete, hence $C \cap [0, \rho]$ is finite, since it is also compact.

We prove Assertion (3). Since there are no critical radius of f within $]r', r''[$, the real valued functions $t \mapsto |\nu^-(t, f)|$ and $t \mapsto |\nu^+(t, f)|$ are locally constant in $]r', r''[$. Hence, by the connectedness of $]r', r''[$, we conclude that $t \mapsto |\nu^-(t, f)|$ and $t \mapsto |\nu^+(t, f)|$ must be constant in $]r', r''[$, which proves item (a). Finally, by item (a), for any $r \in \,]r', r''[$, each multi-index in $\Gamma_r(f)$ has degree $d = |\nu^+(r', f)|$. If $\nu, \nu' \in \Gamma_r(f)$, it is clear that $|a_\nu| = |a_{\nu'}|$. Moreover, if $\gamma \notin \Gamma_r(f)$ is a multi-index of degree d, then $|a_\gamma| < |a_\nu|$. This ends the proof of item (b). □

Proof of Proposition 2.2.3. Assertions (2) and (3) are immediate consequences of Assertion (1), which we prove now. Write $\mu = \nu^+(r, f)$ and $\nu = \nu^+(r, g)$. Let α and β be two multi-indices such that

$$\alpha + \beta = \mu + \nu.$$

If $\mu \prec \alpha$, then we have

$$|a_\alpha b_\beta| < \frac{|f|_r}{r^{|\alpha|}}|b_\beta| \leq \frac{|f|_r}{r^{|\alpha|}}\frac{|g|_r}{r^{|\beta|}} = \frac{|f|_r|g|_r}{r^{|\mu+\nu|}} = |a_\mu b_\nu|.$$

If $\alpha \prec \mu$, then $\nu \prec \beta$ and by a similar argument we obtain $|a_\alpha b_\beta| < |a_\mu b_\nu|$. Thus, we have

$$|c_{\mu+\nu}| = \left| a_\mu b_\nu + \sum_{\substack{\alpha+\beta=\mu+\nu \\ \mu\prec\alpha}} a_\alpha b_\beta + \sum_{\substack{\alpha+\beta=\mu+\nu \\ \nu\prec\beta}} a_\alpha b_\beta \right| = |a_\mu b_\nu| = \frac{|f|_r|g|_r}{r^{|\mu+\nu|}},$$

which proves that $\mu + \nu \in \Gamma_r(fg)$. Let γ be a multi-index satisfying $\mu + \nu \prec \gamma$. If α and β are such that $\gamma = \alpha + \beta$, it follows that $\mu + \nu \prec \alpha + \beta$. If α satisfies $\mu \prec \alpha$ we obtain

$$|a_\alpha b_\beta| < \frac{|fg|_r}{r^{|\gamma|}}$$

and if $\alpha \preceq \mu$, it implies that $\nu \prec \beta$ from we get also

$$|a_\alpha b_\beta| < \frac{|fg|_r}{r^{|\gamma|}}.$$

Thus, we conclude that

$$|c_\gamma| \leq \max_{\alpha+\beta=\gamma} |a_\alpha b_\beta| < \frac{|fg|_r}{r^{|\gamma|}},$$

which implies

$$\nu^+(r, fg) = \nu^+(r, f) + \nu^+(r, g).$$

The other equality is proven in a similar manner. □

2.4 Bounded entire functions and units

The following statements are classical. Some of them will be used later.

Lemma 2.4.1. *An entire function f is bounded if and only if $|f|(r)$ is a bounded real valued function.*

Proof. This follows directly from Proposition 1.2.4 and Lemma 1.1.7. □

Proposition 2.4.2. *A bounded entire function must be constant.*

Proof. Let $f = \sum_\gamma a_\gamma X^\gamma$ be a non-constant entire function. If β is any multi-index of positive degree such that $a_\beta \neq 0$, then

$$|a_\beta| r^{|\beta|} \leq |f|(r)$$

is satisfied by any $r \in \mathbb{R}_{\geq 0}$. Thus, from Lemma 2.4.1 it follows that $|f|(r)$ is unbounded. □

Corollary 2.4.3. *An entire function f for which there is an increasing sequence $(r_i)_{i\in\mathbb{N}}$ of positive numbers converging to $+\infty$, which satisfies*

$$|f|(r_i) \leq \frac{1}{r_i},$$

for each $i \in \mathbb{N}$, must be zero.

Corollary 2.4.4. *The units of the ring $\mathcal{E}_n$ are the non-zero constant functions.*

Proof. First, we notice that $\mathcal{E}_n^\times \subset T_{n,\rho}^\times$ for each $\rho \in |\mathbb{F}^*|$. Given $f \in \mathcal{E}_n^\times$, by Proposition 1.2.2 and by Lemma 1.1.7, it follows that $a_\theta \neq 0$ and that for each $f \in \mathbb{R}_{\geq 0}$

$$|f|(r) = |a_\theta|,$$

which implies that f is bounded. Thus, the assertion follows by lemma 2.4.2. □

2.5 Transcendental entire functions

We recall that an entire function f is called *transcendental* if it is not a polynomial. In the one variable case, an entire function is transcendental exactly when it has infinitely many zeros. In several variables we have a similar statement.

Proposition 2.5.1. *An entire function f is a polynomial if and only if it has finitely many critical radii.*

Proof. Write $f = \sum_\gamma a_\gamma X^\gamma$. If f is a polynomial of degree d, defines β as a multi-index where $\max_{|\gamma|=d} |a_\gamma|$ is reached and set

$$\xi = \max_{|\gamma|<d} \left| \frac{a_\gamma}{a_\beta} \right|^{\frac{1}{|\beta|-|\gamma|}}.$$

Since the set of critical radii of f is contained in the interval $[0, \xi]$, the assertion follows from Proposition 2.2.2.

Assume that f is an entire function with finitely many critical radii. Let $r' > 0$ be the largest critical radius of f and $\nu = \nu^+(r', f)$. Assume there exists a multi-index β whose degree is greater than $|\nu|$ and such that $a_\beta \neq 0$. Set

$$r'' = \left|\frac{a_\nu}{a_\beta}\right|^{\frac{1}{|\beta|-|\nu|}},$$

so that $r'' > r'$ and

$$|a_\beta| r^{|\beta|} > |a_\nu| r^{|\nu|},$$

for each $r > r''$, which contradicts the fact that $\nu = \nu^+(r', f)$. Thus, we conclude that $a_\gamma = 0$ for any γ satisfying $|\gamma| > |\nu|$, which proves that f is a polynomial of degree $|\nu|$. □

Corollary 2.5.2. *An entire function f is transcendental if and only if, for each $d \in \mathbb{N}$, we have*

$$\lim_{r \to +\infty} \frac{|f|(r)}{r^d} = +\infty.$$

Lemma 2.5.3. *Let $P, Q \in \mathbb{F}[X_1, \ldots, X_n]$ be such that* $\deg(P) > \deg(Q)$. *There exists $\rho_0 \in |\mathbb{F}^*|$ such that for any $r > \rho_0$*

$$|P|(r) > |Q|(r).$$

Proof. Write $f = \sum_\gamma a_\gamma X^\gamma$ and $g = \sum_\gamma b_\gamma X^\gamma$. Let ρ' and ρ'' be the largest critical radius of f and g respectively. Put

$$\mu = \nu^+(\rho', f), \qquad \nu = \nu^+(\rho'', g),$$

$$\xi = \left|\frac{b_\nu}{a_\mu}\right|^{\frac{1}{|\mu|-|\nu|}},$$

and

$$\rho_0 = \max\{\xi, \rho', \rho''\}$$

(note that $\deg P = |\mu|$ and $\deg Q = |\nu|$). For each $r > \rho_0$, we have

$$|P|(r) = |a_\mu| r^{|\mu|} > |b_\nu| r^{|\nu|} = |Q|(r),$$

which ends the proof. □

Lemma 2.5.4. *Let f be a transcendental entire function and P be a polynomial. There exists $\rho_0 \in |\mathbb{F}^*|$ such that for each $r > \rho_0$*

$$|f|(r) > |P|(r).$$

Proof. For each $i \in \mathbb{N}$, consider the polynomial

$$f_i = \sum_{|\gamma| \leq i} a_\gamma X^\gamma.$$

Let $j \in \mathbb{N}$ be such that $\deg f_j > \deg P$. By Lemma 2.5.3, there exists $\rho_0 \in |\mathbb{F}^*|$ such that

$$|P|(r) < |f_j|(r) \leq |f|(r),$$

for each $r > \rho_0$. This ends the proof. □

2.6 Partial derivatives

We state the most important fact about partial derivatives of entire and meromorphic functions.

Proposition 2.6.1. *Any entire function is differentiable, its analytic partial derivatives coincide with its formal partial derivatives and they are entire.*

The following result is proven in [ChYe].

Lemma 2.6.2 (Logarithmic Derivative Lemma)**.** *Each meromorphic function f satisfies*

$$|\partial_j f|(r) \leq \frac{1}{r}|f|(r),$$

whenever $r > 0$.

From Lemma 2.6.2 and Lemma 2.4.3, we deduce the following corollary.

Corollary 2.6.3. *Let f be a non-zero entire function. If f divides one of its partial derivatives $\partial_j f$, then $\partial_j f$ is identically zero.*

Given an entire or meromorphic function f, we recall that the n-tuple

$$(\partial_1 f, \ldots, \partial_n f)$$

is denoted by ∇f. We shall write $\nabla f = 0$ when ∇f is the null n-tuple $(0, \ldots, 0)$. It is easily seen that if $\mathbb{F}$ has characteristic zero, a meromorphic function is constant if and only if $\nabla f = 0$. We prove the analogous statement in positive characteristic.

Proposition 2.6.4. *Assume $\mathbb{F}$ has characteristic $p > 0$. An entire function f is a p-th power if and only if $\nabla f = 0$.*

Proof. If f is p-th power, then obviously satisfies $\nabla f = 0$. Reciprocally, assume $\nabla f = 0$ and write $f = \sum_\gamma a_\gamma X^\gamma$. Each $\gamma \in \mathbb{N}^n$ for which $a_\gamma \neq 0$, satisfies

$$\gamma_j \equiv 0 \mod p$$

for $j = 1, \dots, n$. Since $\mathbb{F}$ is algebraically closed, for each γ there exists b_γ such that $b_\gamma^p = a_{p\gamma}$. Set $g = \sum_\gamma b_\gamma X^\gamma$. By Lemma 2.1.1, we obtain

$$f = \sum_\gamma b_\gamma^p X^{p\gamma} = \lim_{i\to\infty} \left(\sum_{|\gamma|\leq i} b_\gamma^p X^{p\gamma} \right) = \lim_{i\to\infty} \left(\sum_{|\gamma|\leq i} b_\gamma X^\gamma \right)^p = g^p$$

which ends the proof. □

2.7 Greatest common divisors and factorization

We recall the following facts which are fundamental in this work.

Proposition 2.7.1. *[Ch, Th 13] The ring $\mathcal{E}_n$ is a greatest common divisor domain. Moreover, given $f_1, \dots, f_k$ in $\mathcal{E}_n$, if g is a greatest common divisor of $f_1, \dots, f_k$ in the ring $\mathcal{E}_n$, then g is a greatest common divisor of $f_1, \dots, f_k$ in $T_{n,\rho}$, for each $\rho \in |\mathbb{F}^*|$.*

Proposition 2.7.2. *[Ch, Th 14] Given a non-zero entire function f, there is an enumerable set $S \subset \mathcal{E}_n$ consisting of irreducible entire functions which are pairwise coprime, and a mapping*

$$\omega\colon S \to \mathbb{N},$$

such that

$$f = \prod_{P\in S} P^{\omega(P)}.$$

Moreover, if S' is any other enumerable set consisting of irreducible entire functions which are pairwise comprime, and for which there is a mapping

$$\omega'\colon S' \to \mathbb{N}$$

satisfying

$$f = \prod_{P\in S'} P^{\omega'(P)},$$

then, there is a bijection $\phi : S \to S'$ such that for each $P \in S$, P and $\phi(P)$ are associated and such that $\omega = \omega' \circ \phi$.

In the previous proposition, if the product is infinite, then its convergence is understood with respect the locally convex topology τ.

Given an irreducible entire function P, Proposition 2.7.2 allow us to define the valuation at P as the map $\mathrm{ord}_P : \mathcal{E}_n^* \to \mathbb{Z}$ given by

$$\mathrm{ord}_P(f) = \max\left\{d\colon \frac{f}{P^d} \in \mathcal{E}_n\right\}$$

Obviously, it satisfies the usual properties of a valuation and it can be extended to the field of meromorphic functions in the usual way.

Definition 2.7.3. Given a non-zero entire function f we define its *support* as the set $\mathrm{supp}(f)$ of its irreducible factors.

Let $p \geq 0$ be the characteristic of $\mathbb{F}$. We recall some basic properties of the order of multiplicity that we will use later.

Lemma 2.7.4. *Let f be a non-zero entire function such that $\partial_j f$ is non-zero for some j, and let $P \in \mathrm{supp}(f)$. If $p = 0$ or p is coprime with $\mathrm{ord}_P(f)$, then*

$$\mathrm{ord}_P(\partial_j f) = \mathrm{ord}_P(f) - 1,$$

Otherwise, we have

$$\mathrm{ord}_P(\partial_j f) \geq \mathrm{ord}_P(f).$$

Proof. Let $d = \mathrm{ord}_P(f)$ and G be an entire function coprime with P which satisfies $f = P^d G$. By computing the j-th partial derivative of f we obtain

$$\partial_j f = P^{d-1}(dG\partial_j P + P\partial_j G).$$

If $\mathbb{F}$ has characteristic 0 or if d is coprime with p, then $dG\partial_j P \neq 0$. By assuming that

$$dG\partial_j P + P\partial_j G$$

is divisible by P, we obtain that P must divide $dG\partial_j P$. Therefore P must divides $\partial_j P$, since G is coprime with P. But Lemma 2.6.3 implies that $\partial_j f$ is zero, a contradiction with our assumption. Therefore, $dG\partial_j P + P\partial_j G$ must be coprime with P. If $\mathbb{F}$ has characteristic $p > 0$ and d is not coprime with p, then $\partial_j f = P^d\partial_j G$, from what we get the relation

$$\mathrm{ord}_P(\partial_j f) = d + \mathrm{ord}_P(\partial_j G).$$

□

With an argument similar to the one in the proof of Lemma 2.7.4, we deduce the following lemma.

Lemma 2.7.5. *Let g and h be two coprime entire functions. Assume $\partial_j g$ is non-zero for some j. Given an irreducible entire function P such that $\mathrm{ord}_P(g) \neq 0$, if $p = 0$ or p is coprime with $\mathrm{ord}_P(g)$, then*

$$\mathrm{ord}_P(h\partial_j g - g\partial_j h) = 0.$$

Corollary 2.7.6. *Let f be a non-zero meromorphic function such that $\partial_j f$ is non-zero for some j, and P be an irreducible entire function such that $\mathrm{ord}_P(f) \neq 0$. If $p = 0$ or p is coprime with $\mathrm{ord}_P(f)$, then*

$$\mathrm{ord}_P(\partial_j f) = \mathrm{ord}_P(f) - 1.$$

Otherwise, we have

$$\mathrm{ord}_P(\partial_j f) \geq \mathrm{ord}_P(f).$$

We also obtain the following corollary of Proposition 2.6.4.

Corollary 2.7.7. *Assume that $\mathbb{F}$ has characteristic $p > 0$. A meromorphic function f is a p-th power if and only if $\nabla f = 0$.*

Chapter 3

Definability in rings of entire functions

3.1 Preliminaries in Logic

In this section, we will briefly recall the necessary basic notions that we need from Logic - see [CoLas] for a general introduction to Logic.

In our context, an algebraic structure is determined by a base-set, a set of distinguished elements (that we may call constants), and for each $m \in \mathbb{Z}_{\geq 1}$, a set of m-ary functions and a set of m-ary relations (think for example of an ordered Abelian group $\langle G; 0, +, \leq \rangle$). In Logic, we are sometimes interested in studying at once all the algebraic structures which have a similar such presentation (the same number of distinguished constants, and for each m, the same number of m-ary functions and relations). This "presentation" is what we call the underlying language of the structure. So for example, the underlying language of an ordered Abelian group $\langle G; 0, +, \leq \rangle$ is the set $\{0, +, \leq\}$.

One can think of it the other way around: given a language $\mathcal{L} = \{0, +, \leq\}$, where the symbol 0 is a *constant symbol*, the symbol + is a *symbol of binary function*, and the symbol $\leq$ is a *symbol of binary relation*, we may consider any algebraic structure over the language $\mathcal{L}$, namely, a strucutre which has one distinguished element, one binary operation, and one binary relation. So, from the point of view of logic, any ordered Abelian group is a structure over $\{0, +, \leq\}$ (or a $\{0, +, \leq\}$-structure), namely, an algebraic structure with a element that stands for (that interprets) the symbol 0, a binary function that interprets the symbol +, and a binary relation that interprets the symbol $\leq$.

More formally, consider a language $\mathcal{L}$. An $\mathcal{L}$-structure is a set M together with

- an element of M for each constant symbol in $\mathcal{L}$,
- a function $M^m \to M$ for each m-ary function symbol in $\mathcal{L}$, and

- a subset of M^m for each m-ary relation symbol in $\mathcal{L}$.

In this thesis, languages will not be assumed to be equalitarian (namely, there may be no symbol for equality), unless otherwise mentioned.

Given a language $\mathcal{L}$, an $\mathcal{L}$-*sentence* is, informally, an expression built from the symbols of $\mathcal{L}$, together with parenthesis, logical conectives, variables and quantifiers, so that the whole thing "makes sense" when interpreted in an $\mathcal{L}$-structure. For example, the expression φ

$$\forall x \exists y (x + y = 0)$$

is a sentence over the language $\{0, +, =\}$, where 0 is a constant symbol, + is a symbol of binary function, and = is a symbol of binary relation.

A sentence may be true or false, depending in which structure it is interpreted. For example, φ holds in $\langle \mathbb{Z}; 0, +, = \rangle$ but not in $\langle \mathbb{N}; 0, +, = \rangle$. When the expression has free variables (namely, variables that are not quantified), we call it an $\mathcal{L}$-formula instead of $\mathcal{L}$-sentence. For example, the expression $\psi(y)$

$$\psi(y) \colon \forall x (x + y = x)$$

is a $\{0, +, =\}$-formula. It *holds* in an Abelian group if and only if y is the neutral element. More generally, given an $\mathcal{L}$-formula φ, we shall write $\varphi(y_1, \ldots, y_m)$ to indicate that φ has all its free variables among $y_1, \ldots, y_m$.

If $\mathfrak{M}$ is an $\mathcal{L}$-structure with base set M and if R is a subset of M^m (namely, an m-ary relation in M), we will say that a formula $\varphi(y_1, \ldots, y_n)$ defines R, or that R is $\mathcal{L}$-definable in $\mathfrak{M}$, if for each $(a_1, \ldots, a_n) \in M^m$ we have: the formula $\varphi(a_1, \ldots, a_n)$ is true in M if and only if $(a_1, \ldots, a_n)$ belongs to R (where $\varphi(a_1, \ldots, a_n)$ is the sentence obtained from $\varphi(y_1, \ldots, y_n)$ after replacing each y_i by a_i). A function is definable if its graph is definable.

Let $\mathcal{L}'$ be the first order language $\mathcal{L} \cup \{\alpha\}$ where α denotes any symbol which is not in $\mathcal{L}$. Given a class $\mathfrak{G}$ of $\mathcal{L}'$-structures, we will say that α is $\mathcal{L}$-*uniformly definable* in the class $\mathfrak{G}$, it there is an $\mathcal{L}$-formula which defines the interpretation of α in each element of $\mathfrak{G}$.

The *elementary theory* (or *full theory*) of an $\mathcal{L}$-structure $\mathfrak{M}$ is the set of sentences of $\mathcal{L}$ which are true in $\mathfrak{M}$. We will say that the elementary theory of an $\mathcal{L}$-structure $\mathfrak{M}$ is undecidable if there is no algorithm (a Turing machine) to decide whether a sentence holds or not in $\mathfrak{M}$.

3.2 Historical context

In 1949, Julia Robinson [Ro] showed that addition + is definable in terms of multiplication and the successor operation S, and that multiplication is definable from addition and divisibility. Namely, the operation $+$ is $\{\cdot, S\}$-definable and multiplication is $\{+, |\}$-definable. Here the languages $\{\cdot, S\}$ and $\{+, |\}$ are assumed to be equalitarian. Actually, she observes that the $\{\cdot, S\}$-definability of addition remains valid if one replaces the semiring of natural numbers by any integral domain with unit. Julia Robinson also asked about the definability of multiplication in terms of the successor operation and the binary relation $\perp$ of coprimess. In 1981, it was proven by Alan Woods [Wo] that this question is equivalent to the same question without equality. He proved that $\{\perp, S\}$-definability of equality, addition, multiplication and the order relation are all equivalent to a classical number-theoretic conjecture, now known as the Erdös-Woods Conjecture. For a general survey on definability in weak arithmetic, see the survey by A. Bès [Bes].

Given an integer x, let us denote by $\operatorname{supp}(x)$ the set of prime divisors of x. The Erdös-Woods Conjecture is the following:

> There exists a positive integer N with the the following property: if x and y are two integers satisfying
>
> $$\operatorname{supp}(x+i) = \operatorname{supp}(y+i)$$
>
> for each $i = 0, \ldots, N$, then $x = y$.

Only conditional results have been obtained so far (see [Wo], [Lan1] and [Lan2]).

In 2002, Maxim Vsemirnov in [Vs] gave an affirmative answer to a polynomial analogue of the Erdös-Woods Conjecture. Vsemirnov deals with polynomial rings over finite fields and with the language

$$\{\perp, S, S_*\}.$$

The symbol S_* is interpreted as the operation

$$S_*(y) = y + X_1,$$

where X_1 denotes the first indeterminate of the polynomial ring. Vsemirnov proves that equality is $\{\perp, S, S_*\}$-definable in the ring $\mathbb{F}_q[X_1, \ldots, X_n]$ and that the elementary theory of

$$\langle \mathbb{F}_q[X_1, \ldots, X_n]; \perp, S, S_* \rangle$$

is undecidable. We prove the analogous statements over rings of ultrametric entire functions, much inspired by the work of Vsemirnov.

3.3 Examples of definable and uniformly definable properties

In order to simplify some statements, throughout this chapter we adopt the following conventions and notation.

1. By an *ultrametric field* we will always mean a complete ultrametric field which is algebraically closed.

2. By a *ring of entire functions* we will always mean a ring of the form $\mathcal{E}_n(\mathbb{F})$, for some ultrametric field $\mathbb{F}$ and some positive integer n (characteristic may be positive or 0).

3. We will denote by Ω the class of all rings of entire functions.

4. We use $\mathscr{L}$ to denote the first order language $\{S, S_*, \perp\}$ where S and S_* are two unary function symbol and $\perp$ is a binary relation symbol.

5. Every ring of entire functions will be considered as a structure over $\mathscr{L}$ where $\perp$ is interpreted as the coprimess relation and the function symbols S and S_* are interpreted as the operations

$$S(x) = x + 1 \quad \text{and} \quad S_*(x) = x + X,$$

   respectively, where X denotes an independent variable of the ring.

6. For each positive integer k, we use $S^k x$ to denote the term of the language $\mathscr{L}$ defined by induction as

$$S^1 x = Sx \qquad \text{and} \qquad S^k x = SS^{k-1}x.$$

   The term $S_*^k x$ is defined analogously.

The proof of the following lemma is classical. We give a proof in order to fix the ideas.

Lemma 3.3.1. *The following properties are uniformly $\{\perp\}$-definable in the class Ω:*

1. *"x is a non-zero constant "*

2. *"x is identically zero",*

Proof. Let $\mathcal{E}$ be any ring in the class Ω. For the first assertion we claim that the $\{\perp\}$-formula

$$\varphi_1(x) = \forall y(x \perp y).$$

works. Indeed, this formula defines the units of the ring which are exactly the non-zero constant functions (see Corollary 2.4.4). For the the second assertion, we consider the $\{\perp\}$-formula

$$\varphi_0(x) = \forall z(\varphi_1(z) \vee \neg(x \perp z)),$$

which obviously is satisfied by 0. If f is non-zero, it is coprime with at least one irreducible entire function. Therefore f does not satisfies φ_0. □

Lemma 3.3.2. *In any ring of entire functions,* 1 *and* X *are* $\mathscr{L}$*-definable.*

Proof. Let $\varphi_0(x)$ be the $\{\perp\}$-formula which defines the property "x is identically zero". It is easily checked that the $\mathscr{L}$-formulas

$$\psi_1(y) = \varphi_0(S^{p-1}y) \qquad \text{and} \qquad \psi_X(z) = \varphi_0(S_*^{p-1}z)$$

define 1 and X, respectively. □

Lemma 3.3.3. *The following classes of sets are uniformly* $\{\perp\}$*-definable in* Ω.

1. $A_{\mathcal{E}} = \{(f,g) \in \mathcal{E}^2 \colon \operatorname{supp}(f) \subseteq \operatorname{supp}(g)\}$,
2. $B_{\mathcal{E}} = \{(f,g) \in \mathcal{E}^2 \colon \operatorname{supp}(f) = \operatorname{supp}(g)\}$,

where $\mathcal{E}$ *run through the class* Ω.

Proof. It is an easy exercise to check that each of the following $\{\perp\}$-formula

1. $\psi_A(x,y) = \forall z (z \perp y \to z \perp x)$,
2. $\psi_B(x,y) = (\psi_A(x,y) \wedge \psi_A(y,x))$,

defines uniformly the respective class. □

3.4 Uniform definability of equality in the class of rings of entire functions

In this subsection, we give a first proof of the AS-BE Theorem for entire functions in several variables (see Theorem 1). From this theorem we deduce Theorem 5, which is the analogue of the Erdös-Woods conjecture for entire functions. The uniform definability of equality is given as an immediate application of Theorem 5 and Lemma 3.3.3.

Fix an ultrametric field $\mathbb{F}$ and consider the ring $\mathcal{E}_n$ of entire functions in n variables $X_1, \dots, X_n$ over $\mathbb{F}$.

Let us introduce the following notation: given two entire functions f and g, we shall write $f \rhd g$ if there exists an increasing sequence $(r_i)_{i \in \mathbb{N}}$ in $\mathbb{R}_{>0}$ which tends to $+\infty$, such that

$$|f|(r_i) \geq |g|(r_i),$$

for any $i \in \mathbb{N}$. We shall write $f \ntriangleright g$ if $f \rhd g$ is not satisfied.

Lemma 3.4.1. *If* f *and* g *are two entire functions such that* $f \ntriangleright g$*, then* $g \rhd f$.

Proof. Let $(r_i)_i$ be any increasing sequence which converges to $+\infty$. By the assumption $f \not\triangleright g$, the increasing sequence $(k_i)_i$ defined by induction as

$$k_0 = \min\{k\colon |f|(r_k) < |g|(r_k)\},$$

$$k_{i+1} = \min\{k > k_i\colon |f|(r_k) < |g|(r_k)\},$$

is well-defined. Thus, the sequence (r_{k_i}) satisfies $|f|(r_{k_i}) < |g|(r_{k_i})$ for each $i \in \mathbb{N}$. $\square$

Lemma 3.4.2. *Let f and g be two entire functions such that $\nabla f \neq 0$ and $f \triangleright g$. If there exists $c \in \mathbb{F}$ such that*

$$\operatorname{supp}(f) = \operatorname{supp}(g) \qquad \text{and} \qquad \operatorname{supp}(f+c) = \operatorname{supp}(g+c),$$

then $f = g$.

Proof. We adapt the proof given in [AdSt] by Adams and Straus for the one variable case. Since $\nabla f \neq 0$, there is j such that $\partial_j f \neq 0$. First we prove that the meromorphic function

$$F = \frac{(f-g)\partial_j f}{f(f+c)}$$

is entire. Let P be an irreducible entire function in $\operatorname{supp}[(f+c)f]$. Since f and $f+c$ are coprime, either $\operatorname{ord}_P(f) = 0$ or $\operatorname{ord}_P(f+c) = 0$. For instance, assume $\operatorname{ord}_P(f) = 0$. Since $\partial_j(f+c) = \partial_j f$, by Lemma 2.7.4, we obtain

$$\begin{aligned} \operatorname{ord}_P(F) &= \operatorname{ord}_P(\partial_j f) + \operatorname{ord}_P(f-g) - \operatorname{ord}_P(f+c) \\ &\geq (\operatorname{ord}_P(f+c) - 1) + \operatorname{ord}_P(f-g) - \operatorname{ord}_P(f+c) \\ &= \operatorname{ord}_P(f-g) - 1. \end{aligned}$$

On the other hand, since $\operatorname{supp}(f+c) = \operatorname{supp}(g+c)$ we have

$$\operatorname{ord}_P(f-g) = \operatorname{ord}_P((f+c)-(g+c)) \geq \min\{\operatorname{ord}_P(f+c), \operatorname{ord}_P(g+c)\} \geq 1,$$

and consequently, $\operatorname{ord}_P(F) \geq 0$, which proves that F is entire.

In order to conclude, we now prove that $F = 0$. For that, we first give an estimation of the maximum term $|F|(r)$. By the strong triangle inequality and the Logarithmic Derivative Lemma (see Lemma 2.6.2), we have

$$|(f-g)\partial_j f|_r \leq \frac{1}{r}|f|_r \max\{|f|_r, |g|_r\}, \tag{3.1}$$

for each $r \in \mathbb{R}_{\geq 0}$. On the other hand, by Proposition 2.4.2, $|f|(r)$ is unbounded. Therefore, since the maximum term of F is increasing function of r, there exists r_0 such that, for any $r \geq r_0$, we have $|f|_r \geq |c|$. Consequently, we obtain

$$|f+c|_r = |f|_r, \tag{3.2}$$

for $r \geq r_0$. Thus, by Relations (3.1) and (3.2) we obtain the estimation

$$|F|(r) \leq \frac{|f|_r \max\{|f|_r, |g|_r\}}{r|f|_r^2} = \frac{\max\{|f|_r, |g|_r\}}{r|f|_r},$$

for $r \geq r_0$. Finally, by hypothesis there is an increasing sequence $(r_i)_i$ in $\mathbb{R}_{\geq 0}$, satisfying $\lim_{i\to\infty} r_i = +\infty$, such that $|f|_{r_i} \geq |g|_{r_i}$ for each $i \in \mathbb{N}$. In particular, for i large enough we have

$$|F|(r_i) \leq \frac{1}{r_i}.$$

Thus, by Lemma 2.4.3 it follows that $F = 0$. □

We can now prove the analogue of the AS-BE Theorem in several variables.

Proof of Theorem 1. Let f and g be two non-constant entire functions over $\mathbb{F}^n$ such that $u(f) = u(g)$. Let $a \neq b \in \mathbb{F}$ be such that $\operatorname{supp}(f - a) = \operatorname{supp}(g - a)$ and $\operatorname{supp}(f - b) = \operatorname{supp}(g - b)$.

Write $m = p^{u(f)}$, so that there exist f_1 and g_1 entire over $\mathbb{F}^n$ such that $f = f_1^m$ and $g = g_1^m$. On the other hand, since $\mathbb{F}$ is algebraically closed, there exist a_1, b_1 in $\mathbb{F}$ such that $a = a_1^m$ and $b = b_1^m$. Note that a_1 and b_1 are distinct. Since

$$\operatorname{supp}(f_1 - a_1) = \operatorname{supp}(f - a) = \operatorname{supp}(g - a) = \operatorname{supp}(g_1 - a_1)$$

and analogously

$$\operatorname{supp}(f_1 - b_1) = \operatorname{supp}(f - b) = \operatorname{supp}(g - b) = \operatorname{supp}(g_1 - b_1),$$

Write $F = f_1 - a_1$, $G = g_1 - a_1$. By Lemma 3.4.1, we may assume that $F \triangleleft G$. Since $u(f_1) = 0$, we have $\nabla F \neq 0$, so $f_1 = g_1$ by applying Lemma 3.4.2 to F and G, taking $c = a_1 - b_1$. We deduce that $f = g$. □

Given $f, g \in \mathcal{E}_n$, we shall say that f and g share a value $b \in \mathbb{F}$ (ignoring multiplicities) if

$$\operatorname{supp}(f - b) = \operatorname{supp}(g - b).$$

With this notion, Theorem 1 can be reformulated as follows

> *Two non-constant entire functions, with the same ramification index, which share two distinct values, are identical.*

Note that when $\mathbb{F}$ has characteristic zero, the condition "with the same ramification index" is empty. As we indicated in the introduction, Corollary 2 states that this condition is necessary in the case of positive characteristic. We now prove this fact.

Proof of Corollary 2. Assume that $\mathbb{F}$ has positive characteristic. Let f and g be two non-constant entire functions over $\mathbb{F}^n$ satisfying $u(f) > u(g)$. Write $k = p^M$, where $M = u(f) - u(g)$. Let a_1 and a_2 be distinct elements in the finite field $\mathbb{F}_{p^{u(g)}}$ which satisfy $\operatorname{supp}(f - a_i) = \operatorname{supp}(g - a_i)$ for $i = 1, 2$. We want to prove that $f = g^k$.

Write

$$s = p^{u(f)}, \quad t = p^{u(g)}, \quad f = h_1^s, \quad \text{and} \quad g = h_2^t$$

for some $h_1, h_2 \in \mathcal{E}_n$. By definition of the ramification index, h_1 and h_2 satisfy

$$u(h_1) = u(h_2) = 0.$$

Since the finite field $\mathbb{F}_{p^{u(g)}}$ is included in both $\mathbb{F}_s$ and $\mathbb{F}_t$, we have $a_i^s = a_i$ and $a_i^t = a_i$ for each $i = 1, 2$. Hence, we obtain

$$\operatorname{supp}(h_1 - a_i) = \operatorname{supp}(f - a_i) = \operatorname{supp}(g - a_i) = \operatorname{supp}(h_2 - a_i),$$

for $i = 1, 2$. Thus, by Theorem 1 it follows that $h_1 = h_2$. Since $s = tk$, we have

$$f = h_1^s = (h_1^t)^k = (h_2^t)^k = g^k.$$

□

Proof of Theorem 5. Let $p \geq 0$ be the characteristic of $\mathbb{F}$. Let f and g be entire functions. Let $a, b \in \mathbb{F}$ and $\ell \in \mathbb{F}[x_1, \ldots, x_n]$ of degree one. Assume that

$$\begin{aligned} \operatorname{supp}(f) &= \operatorname{supp}(g) \\ \operatorname{supp}(f + a) &= \operatorname{supp}(g + a) \\ \operatorname{supp}(f + \ell) &= \operatorname{supp}(g + \ell), \text{and} \\ \operatorname{supp}(f + \ell + b) &= \operatorname{supp}(g + \ell + b). \end{aligned}$$

We want to prove that $f = g$.

Without loss of generality, we may assume $f \triangleright g$ (see Lemma 3.4.1). In the case that $\nabla f \neq 0$, the equality $f = g$ follows from Lemma 3.4.2 because f and g share the values 0 and $-a$. Assume $\nabla f = 0$ and write

$$F = f + \ell \quad \text{and} \quad G = g + \ell.$$

If f is constant, the hypothesis $\operatorname{supp}(f) = \operatorname{supp}(g)$ implies that g is constant. In this case $f = g$ follows by applying Theorem 1 to F and G. Assume that f is non-constant. Since $\nabla f = 0$, this implies $p > 0$ and f is a p-th power. Since f is not a one degree polynomial, by Lemma 2.5.3 and 2.5.4, it follows that

$$|F|(r) = |f|(r)$$

for r large enough.

We prove that g is not a degree one polynomial. Assume it is, so that $F \rhd G$. Since f is a p-th power, $F = f + \ell$ is non constant, and since $\text{supp}(f+\ell) = \text{supp}(g+\ell)$, we deduce that $G = g + \ell$ is non constant. Since

$$\text{supp}(F) = \text{supp}(G) \quad \text{and} \quad \text{supp}(F+b) = \text{supp}(G+b),$$

we deduce that $F = G$, which is impossible.

Since the degree of g is not 1, we deduce, as previously for f, that

$$|G|(r) = |g|(r)$$

for r large enough. Thus, we have $F \rhd G$, and since F and G share the values 0 and $-b$, Lemma 3.4.2 implies $F = G$, hence $f = g$. □

We can now prove that equality is uniformly definable in the class Ω.

Proof of Corollary 6. By Lemma 3.3.3, there is a $\{\perp\}$-formula $\varphi(x, y)$ which uniformly defines the property "$\text{supp}(x) = \text{supp}(y)$" in Ω. Hence, by Theorem 5, the $\mathscr{L}$-formula

$$\varphi_{eq}(x, y) = (\varphi(x, y) \wedge \varphi(Sx, Sy) \wedge \varphi(S_*x, S_*y) \wedge \varphi(SS_*x, SS_*y))$$

uniformly defines the equality in Ω. □

3.5 Proof of Theorem 7

In this section we prove that the elementary theory of the $\mathscr{L}$-structure $\langle \mathcal{E}; \perp, S, S_* \rangle$ is undecidable. We will use $\overline{\mathbb{F}}_p$ to denote the algebraic closure of the finite field $\mathbb{F}_p$. In [Vs], Vsemirnov proves the following statement.

Proposition 3.5.1. *[Vs, Th 3] The elementary theory of the $\mathscr{L}$-structure*

$$\langle \overline{\mathbb{F}}_p[X]; \perp, S, S_* \rangle$$

is undecidable.

In fact, Vsemirnov states the previous proposition for polynomial rings over a finite field, but his proof actually works with no change if the finite field is replaced by its algebraic closure. Throughout this section, $\varphi(x, y)$ and $\psi(x, y)$ are two $\{\perp\}$-formulas which define the properties "$\text{supp}(x) = \text{supp}(y)$" and "$\text{supp}(x) \subseteq \text{supp}(y)$", respectively (see Lemma 3.3.3).

Lemma 3.5.2. *The property "$y = X^{p^m}$ for some $m \in \mathbb{N}^*$" is $\mathscr{L}$-definable in any ring of entire functions of characteristic p.*

Proof. By Corollary 2, the formula

$$\phi_{pw}(y)\colon \neg y = X \wedge \varphi(X, y) \wedge \varphi(SX, Sy)$$

satisfies the assertion. □

Proof of Theorem 7. To prove the undecidability of $\langle \mathcal{E}; \perp, S, S_* \rangle$, it is enough to show that $\overline{\mathbb{F}}_p[X]$ is $\mathscr{L}$-definable in $\mathcal{E}$ and apply Proposition 3.5.1. Indeed, the $\mathscr{L}$-formula

$$\varphi_{pol}(x)\colon x = 0 \vee \exists y[\varphi_{pw}(y) \wedge (\psi(x, S_*^{p-1}y))],$$

which reads in $\mathcal{E}$ as

"$x = 0$ or $\mathrm{supp}(x) \subseteq \mathrm{supp}(X^{p^m} + (p-1)X)$ for some m",

defines the property "x is a polynomimal in $\overline{\mathbb{F}}_p[X]$". This is because, if $f \in \overline{\mathbb{F}}_p[X]$, then there exists $m \in \mathbb{N}^*$ such that all the roots of f are contained in $\mathbb{F}_{p^m}$, namely, f satisfies

$$\mathrm{supp}(f) \subseteq \mathrm{supp}(X^{p^m} - X).$$

□

Chapter 4

Branched Values of Meromorphic functions in several variables

4.1 Divisors of entire functions

Following some ideas from [La] and [EsTuYa], we introduce a group of divisors of entire functions in several variables.

4.1.1 Definitions and properties

Given a non-zero entire function f, we will denote by $[f]$ the equivalence class of f modulo association (e.g.: modulo multiplication by a unit). Since irreducibility is preserved by association, we will denote by $\mathrm{Irr}(\mathcal{E}_n)$ the set of equivalence classes of irreducible entire functions. The elements of $\mathrm{Irr}(\mathcal{E}_n)$ will be called *irreducible points* or simply *points*. Since the map ord, defined by $h \mapsto \mathrm{ord}_h$ on irreducible entire functions, is compatible with association, each point $\mathcal{P} = [h] \in \mathrm{Irr}(\mathcal{E}_n)$ induces a valuation

$$\omega_{\mathcal{P}} = \mathrm{ord}_h$$

on $\mathcal{E}_n$ which extends in a canonical way to the field of meromorphic functions. Given a map

$$T \colon \mathrm{Irr}(\mathcal{E}_n) \to \mathbb{Z},$$

we define its *support* as:

$$\mathrm{Supp}(T) = \{\mathcal{P} \in \mathrm{Irr}(\mathcal{E}_n) \colon T(\mathcal{P}) \neq 0\}.$$

Notation 4.1.1. Given a point $\mathcal{P} \in \mathrm{Irr}(\mathcal{E}_n)$ and a positive real number ρ, we shall write

$$\mathcal{P} \leq \rho$$

if $\mathcal{P}$ is represented by an irreducible entire function P which is not a unit in $T_{n,\rho}$ (namely, which has at least on zero in the dressed polydisk D^n_ρ).

Definition 4.1.2. A *divisor of irreducible points*, or simply a *divisor*, is a mapping

$$T\colon \mathrm{Irr}(\mathcal{E}_n) \to \mathbb{Z}$$

whose support $\mathrm{Supp}(T)$ is such that for each $\rho > 0$, there are at most finitely many points $\mathcal{P} \in \mathrm{Supp}(T)$ satisfying $\mathcal{P} \leq \rho$. The set of divisors of irreducible points will be denoted by

$$\mathscr{D}(\mathcal{E}_n).$$

With the addition $+$ and the order $\geq$ induced by $\mathbb{Z}$, $\mathscr{D}(\mathcal{E}_n)$ is a partially ordered group. A divisor $T \in \mathscr{D}(\mathcal{E}_n)$ is called an *effective divisor* if it satisfies $T \geq 0$. The semigroup of effective divisors is denoted by $\mathscr{D}^+(\mathcal{E}_n)$.

It happens that the support of a divisor is countable (this is Corollary 4.1.6 below). For this reason, in what follows, a divisor T is represented as a formal sum

$$\sum_{i\in J} d_i \mathcal{P}_i$$

where J is a countable set, $\{\mathcal{P}_i\colon i \in J\}$ is a set of irreducible points which contains the support of T, and $d_i = T(\mathcal{P}_i)$.

Let $\mathcal{P}$ be an irreducible point, and $P = \sum a_\gamma X^\gamma$ a function that represents it. Given $r \in \mathbb{R}_{\geq 0}$, we put

$$|\mathcal{P}|_r = \frac{|P|(r)}{|a_{\nu^+(0,P)}|},$$

which is well defined by Proposition 2.2.3. Note that when P is a unit in $T_{n,r}$, the quantity $|\mathcal{P}|_r$ is just 1.

Given a divisor

$$T = \sum_{i\in J} d_i \mathcal{P}_i,$$

for each $r \in \mathbb{R}_{\geq 0}$, we define

$$|T|(r) = \prod_{i\in J} |\mathcal{P}_i|_r^{d_i}.$$

Note that only finitely many terms $|\mathcal{P}_i|_r$ in this product are not equal to one, so that it is indeed a finite product. It is also easily seen that any $T, T' \in \mathscr{D}^+(\mathcal{E}_n)$ satisfy

$$|T + T'|(r) = |T|(r)|T'|(r),$$

whenever $r \in \mathbb{R}_{\geq 0}$.

Finally, given a non-zero entire function, we will write

$$\mathrm{Supp}(f) = \{[h] \in \mathrm{Irr}(\mathcal{E}_n)\colon h \in \mathrm{supp} f\}.$$

We will prove the following lemma and theorem in the next sections.

Lemma 4.1.3. *Let T and T' be two effective divisors. If $T \leq T'$, then there exists a positive real number r' such that $|T|(r) \leq |T'|(r)$, whenever $r \geq r'$.*

Theorem 4.1.4. *1. Given an entire function f, the formal sum*

$$\mathcal{D}(f) = \sum_{\mathcal{P} \in \operatorname{Supp}(f)} \omega_{\mathcal{P}}(f)\mathcal{P}$$

defines an effective divisor. Moreover, writing $f = \sum_\gamma a_\gamma X^\gamma$, we have

$$|\mathcal{D}(f)|(r) = \frac{|f|(r)}{|a_{\nu^+(0,f)}|}.$$

2. For each $f, g \in \mathcal{E}_n^*$, the map

$$\mathcal{D} \colon \mathcal{E}_n^* \to \mathscr{D}^+(\mathcal{E}_n)$$

satisfies

$$\mathcal{D}(f \cdot g) = \mathcal{D}(f) + \mathcal{D}(g).$$

3. For each effective divisor T there is an entire function f which satisfies $T = \mathcal{D}(f)$.

To finish this section, let us note that one can easily extend Theorem 4.1.4 to meromorphic functions.

4.1.2 Proof of Lemma 4.1.3

We first generalize the concept of critical radius to our context. Given an irreducible point $\mathcal{P} = [P]$, we will say that $r \geq 0$ is a critical radius of $\mathcal{P}$ when it is a critical radius of P. Given a divisor T, we will say that $r \geq 0$ is a critical radius of T when r is a critical radius for at least one irreducible point in $\operatorname{Supp}(T)$.

Lemma 4.1.5. *Let T be a divisor.*

1. *For each positive number ρ, the interval $[0, \rho]$ contains only finitely many critical radii of T.*

2. *Each critical radius of T is a critical radius for at most finitely many irreducible points in $\operatorname{Supp}(T)$.*

Proof. Given a positive number r, denote by S_r the set of irreducible points in $\operatorname{Supp}(T)$ with at least one critical radius inside the interval $[0, r]$. The first assertion follows by Lemma 2.2.1, since S_r must be finite by the definition of divisors. For the second assertion, let ρ_0 be a critical radius of T. The set of irreducible points whose set of critical radius contains ρ_0 is contained in S_{ρ_0}. □

Corollary 4.1.6. *The support of an effective divisor is countable.*

Proof. Let T be a divisor and $C \subset \mathbb{R}_{\geq 0}$ be the set of critical radius of T. By the first assertion of Lemma 4.1.5, C is countable. For each $\rho \in C$, denote by $\mathscr{C}_\rho$ the set of irreducible points in $\mathrm{Supp}(T)$ whose first critical radius is ρ. By the second assertion of Lemma 4.1.5, $\mathscr{C}_\rho(T)$ is finite (it could be empty). Thus, the support of T can be written as the union

$$\mathrm{Supp}(T) = \bigcup_{\rho \in C} \mathscr{C}_\rho.$$

□

In this subsection and in the following, it will be useful to introduce the following notation. We will denote by $\mathscr{D}_0^+(\mathcal{E}_n)$ the set of effective divisors whose first critical radius is zero. Analogously, $\mathscr{D}_1^+(\mathcal{E}_n)$ will denote the set of effective divisors whose first critical radius is positive. By the second assertion of Lemma 4.1.5, we observe that divisors of $\mathscr{D}_0^+(\mathcal{E}_n)$ have finite support. It is immediate that for each effective divisor T, there are unique $T_0 \in \mathscr{D}_0^+(\mathcal{E}_n)$ and $T_1 \in \mathscr{D}_1^+(\mathcal{E}_n)$ such that $T = T_0 + T_1$.

Proof of Lemma 4.1.3. Let T and T' be two effective divisors such that $T \leq T'$. We want to find a positive real number r' such that $|T|(r) \leq |T'|(r)$, whenever $r \geq r'$. If any of T and T' is the zero divisor, then the statement is trivial. Assume that both are non-zero and write

$$T = \sum_{i \in J} d_i \mathcal{P}_i \quad \text{and} \quad T' = \sum_{i \in J} d_i' \mathcal{P}_i$$

for some countable set J. We choose r' large enough in order to have $|\mathcal{P}_i|_{r'} \geq 1$, whenever $i \in J$ is such that $\mathcal{P}_i \in \mathscr{D}_0^+(\mathcal{E}_n)$. On the other hand, any irreducible class $\mathcal{P}_i \in \mathscr{D}_1^+(\mathcal{E}_n)$ satisfies $|\mathcal{P}_i|_r \geq 1$ for each $r \in \mathbb{R}_{\geq 0}$. Thus, we obtain

$$|T|(r) = \prod_{i \in J} |\mathcal{P}|_r^{d_i} \leq \prod_{i \in J} |\mathcal{P}|_r^{d_i'} = |T'|(r)$$

for any $r \geq r'$. □

4.1.3 Proof of Theorem 4.1.4

The second assertion of Theorem 4.1.4 is immediate. We prove the first and the third assertions.

Proof of Assertion 1. Consider the map

$$T \colon \mathrm{Irr}(\mathcal{E}_n) \to \mathbb{Z}$$

defined by

$$T(\mathcal{P}) = \omega_{\mathcal{P}}(f)$$

for each $\mathcal{P} \in \operatorname{Irr}(\mathcal{E}_n)$. We only need to check that $\operatorname{Supp}(T)$ is the support of an effective divisor. Write f as the product

$$\prod_{i \in J} P_i^{d_i}$$

of irreducible factors. Let $\rho \in |\mathbb{F}^*|$ be such that $[0, \rho]$ contains at least one critical radius of f. By Proposition 2.2.1, it follows that f is not a unit in $T_{n,\rho}$. Proposition 1.2.3 implies that f is divisible by finitely many irreducible elements $p_1, \ldots, p_k$ in $T_{n,\rho}$. On the other hand, by Proposition 2.7.1, two different irreducible factors P_i and P_j (which are assumed to be coprime if $i \neq j$) must be coprime in $T_{n,\rho}$. Hence, for each p_i there exists only one $j \in J$ such that p_i divides P_j in the ring $T_{n,\rho}$. Thus, it follows that only finitely many entire functions P_j are non-units in $T_{n,\rho}$, which concludes the proof. □

In order to prove the third assertion of Theorem 4.1.4, we first give some criteria for the convergence of infinite products of entire functions. Given a sequence $(f_i)_i$ of elements of $\mathcal{E}_n$, we recall that the infinite product

$$\prod_{i=0}^{\infty} f_i$$

converges to $f \in \mathcal{E}_n$ with respect the locally convex topology τ if it converges to f in $(T_{n,\rho}, |\ \ |_\rho)$, for each $\rho \in |\mathbb{F}^*|$. The following proposition follows directly from Lemma 1.1.5.

Lemma 4.1.7 (Proposition 2.1, [ChYe]). *If $(f_i)_i$ is a sequence of entire functions which satisfies*

$$\lim_{i \to \infty} |f_i - 1|_\rho = 0,$$

for each $\rho \in |\mathbb{F}^*|$, then the infinite product $\prod_{i=1}^{\infty} f_i$ converges to an entire function f.

Lemma 4.1.8. *If (f_i) is a sequence of entire functions with constant term equal to 1, and for which there exists an increasing sequence $(\rho_i)_{i \in \mathbb{N}}$ of elements in $|\mathbb{F}^*|$ which converges to $+\infty$, and such that each $f_i \in T^{\times}_{n,\rho_i}$, then the infinite product $\prod_{i=1}^{\infty} f_i$ is convergent in $\mathcal{E}_n$.*

Proof. For each $j \in \mathbb{N}$, if we write

$$f_j = 1 + \sum_{|\gamma| \geq 1} a_{j,\gamma} X^\gamma,$$

by Proposition 1.2.2, we obtain that

$$|a_{j,\gamma}| \rho_j^{|\gamma|} \leq 1, \quad \text{if } |\gamma| \geq 1, \tag{4.1}$$

since $f_j \in T^{\times}_{n,\rho_j}$ by assumption. From Equation (4.1), it follows that for each $i \in \mathbb{N}$ and any $j > i$, we have

$$|1 - f_j|_{\rho_i} = \max_{|\gamma| \geq 1} |a_{j,\gamma}| \rho_i^{|\gamma|} \leq \max_{|\gamma| \geq 1} \left(\frac{\rho_i}{\rho_j} \right)^{|\gamma|} \leq \frac{\rho_i}{\rho_j},$$

from what we get

$$0 \leq \lim_{j \to \infty} |1 - f_j|_{\rho_i} \leq \lim_{j \to \infty} \frac{\rho_i}{\rho_j} = 0.$$

Since, for each $\rho \in |\mathbb{F}^*|$ there exists $i \in \mathbb{Z}_+$ such that $\rho \leq \rho_i$, we obtain that

$$0 \leq |1 - f|_{\rho} \leq |1 - f|_{\rho_i} = 0.$$

Thus, the assertion follows from Lemma 4.1.7. □

Lemma 4.1.9. *If $(f_i)_i$ is a sequence of non-constant entire functions with constant term equal to 1 such that for each $\rho \in |\mathbb{F}^*|$, the set*

$$T_{n,\rho} \setminus T^{\times}_{n,\rho}$$

contains only finitely many elements of the sequence, then the infinite product $\prod_{i=1}^{\infty} f_i$ converges in $\mathcal{E}_n$.

Proof. Given a non-zero entire function f, denote by $c(f)$ the first critical radius of f. By hypothesis, the set of real numbers

$$\{c(f_i) \colon i \in \mathbb{N}\}$$

defines an increasing sequence (r_j) of elements in $|\mathbb{F}^*|$ which converges to $+\infty$. For each $j \in \mathbb{N}$, define the entire function h_j as the finite product

$$h_j = \prod_{c(f_i) = r_j} f_i.$$

Thus we obtain a sequence of entire functions $(h_j)_j$ such that the first critical radius of h_j is r_j. Since $r_0 > 0$, we may choose a sequence ρ_j of elements in $|\mathbb{F}^*|$ which satisfies

$$0 < \rho_0 < r_0 \quad \text{and} \quad r_{i-1} < \rho_i < r_i,$$

for each $j > 1$. It is immediate that $(\rho_j)_j$ converges to $+\infty$. Therefore, $(h_j)_j$ is a sequence of entire functions whose constant term is equal to 1 and such that $h_j \in T^{\times}_{n,\rho_j}$, for each j. Hence, by Lemma 4.1.8, it follows that

$$\prod h_j = \prod f_i$$

converges in $\mathcal{E}_n$. □

Proof of Assertion 3. Let T be an effective divisor. If $\mathrm{Supp}(T)$ is finite, the assertion is trivial. Assume $\mathrm{Supp}(T)$ is infinite. There exists $T_0 \in \mathscr{D}_0^+(\mathcal{E}_n)$ and $T_1 \in \mathscr{D}_1^+(\mathcal{E}_n)$ such that $T = T_0 + T_1$. Since the support of T_0 is finite, it is easy to construct an entire function f_0 satisfying $T_0 = \mathcal{D}(f_0)$. For each $\mathcal{P} \in \mathrm{Supp}(T_1)$, we may choose an entire functions $f_{\mathcal{P}} \in \mathcal{P}$ with constant term equal to 1. Since the sequence

$$(f_{\mathcal{P}})_{\mathcal{P}\in\mathrm{Supp}(T_1)}$$

satisfies the hypothesis of Lemma 4.1.9, the entire function

$$f_1 = \prod_{\mathcal{P}\in\mathrm{Supp}(T)} f_{\mathcal{P}}^{T(\mathcal{P})}$$

satisfies $T_1 = \mathcal{D}(f_1)$. Thus, we have $T = \mathcal{D}(f_0 \cdot f_1)$ which ends the proof. □

4.2 Nevanlinna Theory in several variables

We start with generalizing the concept of *counting function of zeros* to the several variable case. Our definition turns out to be equivalent to that given in [ChYe], though we get to it in a very different manner. Finally, let us note that in the one variable case, this definition is valid also for analytic functions in stripped balls.

Definition 4.2.1. Let f be a non-zero entire function. For each positive number r, denote by $\mathcal{C}_r(f)$ the set of critical radii of f contained in the interval $]0, r]$. The *counting function of zeros of f* is defined as

$$Z(r, f) = |\nu^+(0, f)| \log r + \sum_{t\in\mathcal{C}_r(f)} (|\nu^+(t, f)| - |\nu^-(t, f)|) \log \frac{r}{t}.$$

We will prove the following proposition in the next section.

Proposition 4.2.2 (Poisson-Jensen Formula)**.** *Let f be a non-zero entire function. Write $f = \sum_\gamma a_\gamma X^\gamma$. For each $r > 0$, we have*

$$\log |f|(r) - \log |a_{\nu^+(0,f)}| = Z(r, f).$$

From the Poisson-Jensen Formula and Theorem 4.1.4, it is immediate that each non-zero entire function satisfies

$$Z(r, f) = \log(|\mathcal{D}(f)|(r)). \tag{4.2}$$

For this reason, given an effective divisor T and an entire function f such that $T = \mathcal{D}(f)$, we will write

$$Z(r, T) = Z(r, f).$$

The following corollary is an easy consequence of the discussion above, and the fact that $Z(r, \cdot)$ transforms products into sums (by Equation (4.2), since $|\mathcal{D}(\cdot)|(r)$ transforms products into products).

Corollary 4.2.3. *Let f be a non-zero entire function. For each $r > 0$, we have*

$$Z(r, f) = \sum_{\mathcal{P} \in \mathrm{Supp}(f)} \omega_{\mathcal{P}}(f) Z(r, \mathcal{P}).$$

For convenience of the reader, we recall some notation that are especially relevant to this section.

Notation 4.2.4.

1. The characteristic of $\mathbb{F}$ will be denoted by p and the characteristic exponent by q, i.e. $q = p$ if $p > 0$, and $q = 1$ if $p = 0$.

2. Let f be a non-constant meromorphic function. If $p > 0$, we will denote by $u(f)$ the *ramification index of f* which is defined as the largest positive integer ℓ such that f is a p^{ℓ}-th power. If $p = 0$, we set $u(f) = 0$. The fact that this notion is well defined is a straightforward adaptation of the fact that it is well defined in the one variable case - see [BoEs].

Definition 4.2.5. Let f be a non-zero entire function. Denote by $\Delta(f)$ the set of irreducible points

$$\{\mathcal{P} \in \mathrm{Supp}(f) \colon q^{u(f)+1} \nmid \omega_{\mathcal{P}}(f)\}.$$

The *counting function of zeros of f without multiplicities* is defined as

$$\overline{Z}(r, f) = \sum_{\mathcal{P} \in \Delta(f)} Z(r, \mathcal{P}).$$

We also introduce the following divisor associated to f:

$$\overline{\mathcal{D}}(f) = \sum_{\mathcal{P} \in \Delta(f)} \mathcal{P}.$$

Recall that, given a non-zero meromorphic function f, the function of zeros of f is denoted by f_0 and its function of poles by f_∞, so that $f = f_0/f_\infty$. The set of irreducible points $\mathrm{Supp}(f_0)$ will be called the *support of zeros of f* and will be denoted by $\mathrm{Supp}(f)$. Analogously, the *support of poles of f* is the set $\mathrm{Supp}(f_\infty)$ and will be denoted by $\mathrm{Supp}_\infty(f)$.

Definition 4.2.6. Let f be a non-zero meromorphic function.

1. The *counting function of zeros of f with multiplicities* is defined as

$$Z(r, f) = Z(r, f_0).$$

2. The *counting function of zeros of f without multiplicities* is defined as

$$\overline{Z}(r, f) = \overline{Z}(r, f_0).$$

3. The *counting functions of poles of f with multiplicities* is defined as

$$N(r,f) = Z(r,f_\infty).$$

4. The *counting functions of poles of f without multiplicities* is defined as

$$\overline{N}(r,f) = \overline{Z}(r,f_\infty).$$

5. The *Nevanlinna function of f* is defined as

$$T(r,f) = \max\{Z(r,f), N(r,f)\}.$$

6. If $\mathcal{P}$ is a point and C is a subset of $\mathbb{F}$, we may write $f(\mathcal{P}) \in C$ to mean that there exists $b \in C$ such that $\mathcal{P} \in \mathrm{Supp}(f-b)$. It is also convenient to denote b by $f(\mathcal{P})$.

7. Assume $\partial_j f$ is non-zero for some j. Let $S = \{a_1, \dots, a_m\} \subset \mathbb{F}$ be a finite set of m distinct values. We define the following sets of irreducible points

$$A_j^S(f) = \mathrm{Supp}(\partial_j f) \setminus \bigcup_{i=1}^{m} \mathrm{Supp}(f - a_i),$$

$$B_j^S(f) = \{\mathcal{P} \in \mathrm{Supp}(\partial_j f) \colon f(\mathcal{P}) \in S \text{ and } p \mid \omega_{\mathcal{P}}(f - f(\mathcal{P}))\},$$

and

$$\Xi(f) = \{\mathcal{P} \in \mathrm{Supp}_\infty(f) \colon p \mid \omega_{\mathcal{P}}(f)\}.$$

In particular, in characteristic 0, both $B_j^S(f)$ and $\Xi(f)$ are empty. Set

$$Z_0^S(r,\partial_j f) = \sum_{\mathcal{P} \in A_j^S(f)} \omega_{\mathcal{P}}(\partial_j f) Z(r,\mathcal{P}) + \sum_{\mathcal{P} \in B_j^S(f)} (\omega_{\mathcal{P}}(\partial_j f) - \omega_{\mathcal{P}}(f - f(\mathcal{P})) Z(r,\mathcal{P})$$

and

$$N_0(r,\partial_j f) = \sum_{\mathcal{P} \in \Xi(f)} (\omega_{\mathcal{P}}(\partial_j f) - \omega_{\mathcal{P}}(f)) Z(r,\mathcal{P}).$$

Given a non-zero meromorphic function f, whose reduced decomposition is f_0/f_∞, we shall say that a positive number ρ is a *critical radius of f* if ρ is a critical radius of f_0 or f_∞.

Notation 4.2.7. 1. Given $f_1, \dots, f_m \in \mathcal{M}_n^*$, we will denote by

$$\mathscr{C}(f_1, \dots, f_m)$$

the open interval $]0,\rho[$, where ρ is the first positive critical radius of the product $f_1 \dots f_m$.

2. Given three real valued functions ξ, ζ and η, defined in an interval $J =]r_0, \infty[$, we shall write
$$\xi(r) \leq \zeta(r) + O(\eta(r))$$
if there exists a constant $C \in \mathbb{R}$ such that $\xi(r) \leq \zeta(r) + C\eta(r)$ (if η is constant, we may just say that $\xi - \zeta$ is *bounded from above*). Also, we shall write
$$\xi(r) = \zeta(r) + O(\eta(r))$$
if there is a constant $M \in \mathbb{R}_{>0}$ such that $|\xi(r) - \zeta(r)| \leq M\eta(r)$.

Theorem 4.2.8 (Nevanlinna's Second Fundamental Theorem)**.** *Let f be a non-constant meromorphic function, and $a_1, \ldots, a_m$ be pairwise distinct elements of $\mathbb{F}$, where $m \geq 2$. Consider the set*
$$S = \{b_1, \ldots, b_m\},$$
where each b_i is a $p^{u(f)}$-th root of a_i. Let g be a $p^{u(f)}$-th root of f, and let j be such that $\partial_j g \neq 0$. Fix $\rho_0 \in \mathscr{C}(f - a_1, \ldots, f - a_m, g, \partial_j g)$. For any $r \in]\rho_0, +\infty[$, we have
$$\frac{m-1}{q^{u(f)}} T(r, f) \leq \sum_{i=1}^{m} \overline{Z}(r, f - a_i) + \overline{N}(r, f) - Z_0^S(r, \partial_j g) - N_0^S(r, \partial_j g) - \log r + O(1).$$

In practive, we will use the following corollary, which is an immediate consequence of Theorem 4.2.8.

Corollary 4.2.9. *Let f be a non-constant meromorphic function and $a_1, \ldots, a_m$ be pairwise distinct elements of $\mathbb{F}$, where $m \geq 2$. For r large enough, we have*
$$\frac{m-1}{q^{u(f)}} T(r, f) \leq \sum_{i=1}^{m} \overline{Z}(r, f - a_i) + \overline{N}(r, f) - \log r + O(1).$$

4.2.1 Proof of Proposition 4.2.2

Lemma 4.2.10. *Let f be a non-zero entire function. If r_1 and r_2 are two positive numbers such that the open interval $]r_1, r_2[$ does not contain any critical radius of f, then we have*
$$\frac{|f|(r_2)}{|f|(r_1)} = \left(\frac{r_2}{r_1}\right)^{\nu^+(r_1, f)}.$$

Proof. By the third assertion of Proposition 2.2.2, for each $r \in [r_1, r_2[$ (and in particular for $r = r_1$), the maximum term is given by
$$|f|(r) = |a_{\nu^+(r_1,f)}| r^{|\nu^+(r_1,f)|}.$$

On the other hand, the continuity of the maximum term implies that

$$|f|(r_2) = \lim_{r\to r_2^-} |f|(r) = |a_{\nu^+(r_1,f)}| r_2^{\nu^+(r_1,f)}.$$

□

Proof of Proposition 4.2.2. Let $\rho_0 \in \mathscr{C}(f)$ and $\rho_1 < \cdots < \rho_m$ be the critical radii of f inside the interval $]0,r[$. Set $\rho_{m+1} = r$, and for each $i = 0,\ldots,m+1$, write $\nu_i = \nu^+(\rho_i, f)$. It must be noted that $\nu_0 = \nu^+(0,f)$, since f has no critical radius inside $]0,\rho_0[$. So, by Lemma 4.2.10, we obtain

$$\frac{|f|(r)}{|f|(\rho_0)} = \prod_{i=0}^{m} \frac{|f|(\rho_{i+1})}{|f|(\rho_i)} = \prod_{i=0}^{m} \left(\frac{\rho_{i+1}}{\rho_i}\right)^{|\nu_i|}.$$

On the other hand, we have

$$\begin{aligned}
\prod_{i=0}^{m} \left(\frac{\rho_{i+1}}{\rho_i}\right)^{|\nu_i|} &= \frac{\rho_{m+1}^{|\nu_m|}}{\rho_0^{|\nu_0|}} \prod_{i=1}^{m} \left(\frac{1}{\rho_i}\right)^{|\nu_i|-|\nu_{i-1}|} \\
&= \left(\frac{\rho_{m+1}}{\rho_0}\right)^{|\nu_0|} \frac{\rho_{m+1}^{|\nu_m|}}{\rho_{m+1}^{|\nu_0|}} \prod_{i=1}^{m} \left(\frac{1}{\rho_i}\right)^{|\nu_i|-|\nu_{i-1}|} \\
&= \left(\frac{\rho_{m+1}}{\rho_0}\right)^{|\nu_0|} \prod_{i=1}^{m} \rho_{m+1}^{|\nu_i|-|\nu_{i-1}|} \prod_{i=1}^{m} \left(\frac{1}{\rho_i}\right)^{|\nu_i|-|\nu_{i-1}|} \\
&= \left(\frac{\rho_{m+1}}{\rho_0}\right)^{|\nu_0|} \prod_{i=1}^{m} \left(\frac{\rho_{m+1}}{\rho_i}\right)^{|\nu_i|-|\nu_{i-1}|}.
\end{aligned}$$

Thus, we have

$$\frac{|f|_r}{|f|_{\rho_0}} = \left(\frac{r}{\rho_0}\right)^{|\nu_0|} \prod_{i=1}^{m} \left(\frac{r}{\rho_i}\right)^{|\nu_i|-|\nu_{i-1}|}$$

which is equivalent to

$$\log|f|_r - \log|f|_{\rho_0} = |\nu_0|(\log r - \log\rho_0) + \sum_{i=1}^{m}(|\nu_i| - |\nu_{i-1}|)\log\left(\frac{r}{\rho_i}\right).$$

Therefore, we have

$$\begin{aligned}
Z(r,f) &= |\nu_0|\log r + \sum_{i=1}^{m}(|\nu_i| - |\nu_{i-1}|)\log\left(\frac{r}{\rho_i}\right) \\
&= \log|f|_r - \log|f|_{\rho_0} + |\nu_0|\log\rho_0 \\
&= \log|f|_r - \log(|a_{\nu_0}|\rho_0^{|\nu_0|}) + |\nu_0|\log\rho_0 \\
&= \log|f|_r - \log|a_{\nu_0}|,
\end{aligned}$$

where the first equality comes from the definition of $Z(r,f)$. □

4.2.2 Proof of Theorem 4.2.8

In this subsection, we prove several versions of the Nevanlinna's Second Fundamental Theorem.

Lemma 4.2.11. *Let* $f, g \in \mathcal{E}_n^*$ *and fix* $\rho_0 \in \mathscr{C}(f, g, f+g)$. *For each* $r \in]\rho_0, +\infty[$ *we have*

1. $Z(r, fg) = Z(r, f) + Z(r, g)$ *and*
2. $Z(r, f+g) \leq \max\{Z(r, f), Z(r, g)\} + O(1)$.

Proof. The first assertion is an immediate application of Proposition 4.2.2 (Poisson-Jensen Formula). For the second assertion, write

$$f = \sum_{\gamma} a_\gamma X^\gamma \quad \text{and} \quad g = \sum_{\gamma} b_\gamma X^\gamma.$$

By Proposition 4.2.2 applied to $f+g$ and the strong triangle inequality, we obtain

$$Z(r, f+g) = \log|f+g|_r + K \leq \max\{\log|f|_r, \log|g|_r\} + K,$$

where

$$K = -\log|a_{\nu^+(0,f)} + b_{\nu^+(0,g)}|.$$

On the other hand, we have

$$\log|f|_r = Z(r, f) + \log|a_{\nu^+(\rho_0,f)}|$$

and

$$\log|g|_r = Z(r, g) + \log|b_{\nu^+(\rho_0,g)}|.$$

If

$$M = \max\{\log|a_{\nu^+(\rho_0,f)}|, \log|b_{\nu^+(\rho_0,g)}|\},$$

then we obtain

$$\log|f|_r \leq Z(r, f) + M$$

and

$$\log|g|_r \leq Z(r, g) + M$$

Thus, it has been proven that there is a constant C such that

$$Z(r, f+g) \leq \max\{Z(r, f), Z(r, g)\} + C,$$

which proves the second assertion. □

Theorem 4.2.12 (Nevanlinna's First Fundamental Theorem)**.** *Let f be a non-zero meromorphic function and fix $\rho_0 \in \mathscr{C}(f)$. For each $r \in]\rho_0, +\infty[$, we have*

$$\log |f|_r = Z(r, f) - N(r, f) + O(1).$$

Proof. Let f_0 and f_∞ be the function of zeros and poles of f, respectively. Write $f_0 = \sum_\gamma a_\gamma X^\gamma$ and $f_\infty = \sum_\gamma b_\gamma X^\gamma$. Then we have

$$\log |f|(r) = \log |\mathcal{D}(f_0)|(r) - \log |\mathcal{D}(f_\infty)|(r) + \log \left| \frac{a_{\nu^+(0,f_0)}}{b_{\nu^+(0,f_\infty)}} \right|.$$

□

Proposition 4.2.13. *Let f and g be two non-zero meromorphic functions and let b be a constant. Fix $\rho_0 \in \mathscr{C}(f, g, f+g, f-b)$. For each $r \in]\rho_0, +\infty[$ we have*

1. $Z(r, fg) \leq Z(r, f) + Z(r, g) + O(1)$,
2. $N(r, fg) \leq N(r, f) + N(r, g) + O(1)$,
3. $T(r, fg) \leq T(r, f) + T(r, g) + O(1)$,
4. $Z(r, f+g) \leq \max\{Z(r, f) + N(r, g), N(r, f) + Z(r, g)\} + O(1)$,
5. $N(r, f+g) \leq N(r, f) + N(r, g) + O(1)$,
6. $T(r, f+g) \leq T(r, f) + T(r, g) + O(1)$,
7. $T(r, f-b) = T(r, f) + O(1)$, *and*
8. *if $f \in \mathcal{E}_n$, then $T(r, f) = Z(r, f) + O(1)$,*

whenever $r \in]\rho_0, +\infty[$.

Proof. Let f_0, f_∞, g_0 and g_∞ be the respective functions of zeros and poles of f and g. By Lemma 4.1.3 and Lemma 4.2.11, we have

$$Z(r, fg) \leq Z(r, f_0 g_0) + O(1) = Z(r, f_0) + Z(r, g_0) + O(1) = Z(r, f) + Z(r, g) + O(1),$$

and similarly

$$N(r, fg) \leq Z(r, f_\infty g_\infty) + O(1) = Z(r, f_\infty) + Z(r, g_\infty) + O(1) = N(r, f) + N(r, g) + O(1).$$

This proves the first two assertions of the proposition.

Now we prove the third assertion. By applying Assertions 1 and 2 we obtain

$$Z(r, fg) \leq Z(r, f) + Z(r, g) + O(1) \leq T(r, f) + T(r, g) + O(1),$$

and

$$N(r, fg) \leq N(r, f) + N(r, g) + O(1) \leq T(r, f) + T(r, g) + O(1),$$

from what we get

$$T(r, fg) = \max\{Z(r, fg), N(r, fg)\} \leq T(r, f) + T(r, g) + O(1).$$

The fourth assertion follows from the following relations:

$$\begin{aligned} Z(r, f + g) \leq & Z(r, f_0 g_\infty + f_\infty g_0) \\ \leq & \max\{Z(r, f_0 g_\infty), Z(r, f_\infty g_0)\} + O(1) \\ = & \max\{Z(r, f_0) + Z(r, g_\infty), Z(r, f_\infty) + Z(r, g_0)\} + O(1) \\ = & \max\{Z(r, f) + N(r, g), N(r, f) + Z(r, g)\} + O(1). \end{aligned}$$

Assertion 5 follows from the following inequality:

$$N(r, f+g) \leq Z(r, f_\infty g_\infty) + O(1) = Z(r, f_\infty) + Z(r, g_\infty) + O(1) = N(r, f) + N(r, g) + O(1).$$

Now we prove Assertion 6. By Assertions 4 and 5 we have

$$Z(r, f + g) \leq T(r, f) + T(r, g) + O(1) \quad \text{and} \quad N(r, f + g) \leq T(r, f) + T(r, g),$$

and henceforth

$$T(r, f + g) = \max\{Z(r, f + g), N(r, f + g)\} \leq T(r, f) + T(r, g) + O(1).$$

Assertion 7 is immediate from Assertion 6 and the fact that $T(r, b) = 0$.
Assertion 8 is a simple consequence of Proposition 4.2.2. □

As a direct application of the fourth assertion of Proposition 4.2.13, we obtain that for any meromorphic function f and for any value $c \in \mathbb{F}$, the difference

$$Z(r, f - c) - T(r, f)$$

is bounded from above. The following lemma states that, there is at most one value $b \in \mathbb{F}$ for which the difference

$$T(r, f) - Z(r, f - b)$$

is not bounded from above.

Lemma 4.2.14. *Let f be a non-constant meromorphic function. Suppose there exists $b \in \mathbb{F}$ and a sequence of intervals $I_k = [u_k, v_k]$ which satisfies*

$$u_k < v_k < u_{k+1} \qquad \text{and} \qquad \lim_{k\to\infty} u_n = +\infty$$

for each $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty}\left(\inf_{r\in I_k}(T(r,f)-Z(r,f-b))\right)=+\infty.$$

If $c \in \mathbb{F}$ is not equal to b, then

$$Z(r,f-c)=T(r,f)+O(1)$$

is satisfied by any $r \in I_k$, for k large enough.

Proof. Write $F = f - b$, so that

$$\lim_{k\to\infty}\left(\inf_{r\in I_k}(T(r,F+b)-Z(r,F))\right)=+\infty.$$

Since

$$T(r,F+b)=T(r,F)+O(1),$$

it follows that

$$\lim_{k\to\infty}\left(\inf_{r\in I_k}(T(r,F)-Z(r,F))\right)=+\infty.$$

Thus, we have $T(r,F) = N(r,F)$ for each $r \in I_k$, as long as k is large enough. Hence, if F_0 and F_∞ are respectively the functions of zeros and poles of F, then we have

$$\lim_{k\to\infty}\left(\inf_{r\in I_k}(Z(r,F_\infty)-Z(r,F_0))\right)=+\infty. \tag{4.3}$$

Given any constant c which is not equal to b, we put $a = c - b \neq 0$. So we have

$$Z(r,F-a)=Z(r,F_0-aF_\infty)=\log|F_0-aF_\infty|_r+O(1)$$

On the other hand, we have

$$Z(r,F_\infty)=Z(r,aF_\infty)=\log|aF_\infty|_r+O(1)$$

and

$$Z(r,F_0)=\log|F_0|_r+O(1),$$

so from Equation (4.3), it follows that for each $r \in I_k$ with k large enough, we have

$$|aF_\infty|_r>|F_0|_r.$$

Thus, we obtain

$$Z(r,F-a)=\log|aF_\infty|_r+O(1)=Z(r,aF_\infty)+O(1)=T(r,F)+O(1),$$

for $r \in I_k$ and k large enough. Finally, since

$$f - c = (f - b) - (c - b) = F - a$$

it follows that

$$Z(r, f - c) = T(r, f - b) + O(1) = T(r, f) + O(1),$$

for $r \in I_k$ and k large enough, which ends the proof. □

The following corollary is sometimes seen as a weak version of the Second Fundamental Theorem.

Corollary 4.2.15. *Let f be a meromorphic function, $m \geq 2$ an integer, and $a_1, \ldots, a_m$ pairwise distinct elements of $\mathbb{F}$. Fix $\rho_0 \in \mathscr{C}(f - a_1, \ldots, f - a_m)$. For each $r \in]\rho_0, +\infty[$ we have*

$$(m-1)T(r, f) \leq \max_{1 \leq j \leq m} \left(\sum_{i=1, i \neq j}^{m} Z(r, f - a_i) \right) + O(1).$$

Proof. For each $j = 1, \ldots, m$, consider the real valued function

$$\phi_j(r) = \sum_{i=1, i \neq j}^{m} Z(r, f - a_i).$$

Since for each j we have (trivially)

$$\sum_{i=1, i \neq j}^{m} (T(r, f) - Z(r, f - a_i)) \geq (m-1)T(r, f) - \max_{1 \leq \ell \leq m} \phi_\ell(r),$$

if the difference

$$(m-1)T(r, f) - \max_{1 \leq \ell \leq m} \phi_\ell(r)$$

is assumed to be unbounded from above, then for each j, there exists $k_j \neq j$ such that

$$T(r, f) - Z(r, f - a_{k_j})$$

is unbounded from above. In particular, Since $m \geq 2$, this implies that there are two different values a_k and $a_{k'}$ such that

$$T(r, f) - Z(r, f - a_k)$$

and

$$T(r, f) - Z(r, f - a_{k'})$$

are unbounded from above, which is a contradiction with the assertion of Lemma 4.2.14. □

Lemma 4.2.16. *Let $S = \{a_1, \dots, a_m\}$ be a set of m distinct elements of $\mathbb{F}$. If f is a meromorphic function whose j-th partial derivative $\partial_j f$ is non-zero, then*

$$\sum_{i=1}^{m}(Z(r, f-a_i) - \overline{Z}(r, f-a_i)) = Z(r, \partial_j f) - Z_0^S(r, \partial_j f).$$

Moreover, we have

$$N(r, \partial_j f) \leq N(r, f) + \overline{N}(r, f) - N_0^S(r, \partial_j f) + O(1).$$

Proof. This comes immediately from the definitions and Corollary 2.7.6. □

Lemma 4.2.17. *Given a non-zero meromorphic function f such that $\partial_j f \neq 0$ for some j, we have*

$$Z(r, \partial_j f) \leq Z(r, f) + \overline{N}(r, f) - N_0^S(r, \partial_j f) - \log r + O(1)$$

for each $r \in]\rho_0, +\infty[$, where $\rho_0 \in \mathscr{C}(f, \partial_j f)$.

Proof. By Theorem 4.2.12 (Nevanlinna's First Fundamental Theorem) applied to f and then to $\partial_j f$, we have

$$\log |f|_r = Z(r, f) - N(r, f) + O(1)$$

and

$$\log |\partial_j f|_r = Z(r, \partial_j f) - N(r, \partial_j f) + O(1),$$

for each $r \in]\rho_0, +\infty[$, where $\rho_0 \in \mathscr{C}(f, \partial_j f)$.

Therefore, from Lemma 2.6.2 (Logarithmic Derivative Lemma) we obtain

$$Z(r, \partial_j f) - N(r, \partial_j f) \leq Z(r, f) - N(r, f) - \log r + O(1).$$

We conclude the proof by applying the second assertion of Lemma 4.2.16. □

$$\frac{m-1}{q^{u(f)}} T(r, f) \leq \sum_{i=1}^{m} \overline{Z}(r, f-a_i) + \overline{N}(r, f) - Z_0^S(r, \partial_j g) - N_0^S(r, \partial_j g) - \log r + O(1).$$

Proof of Theorem 4.2.8. Fix a non-constant meromorphic function f, an integer $m \geq 2$, and pairwise distinct elements $a_1, \dots, a_m$ of $\mathbb{F}$. For each i, let b_i be a $p^{u(f)}$-th root of a_i, $S = \{b_1, \dots, b_m\}$, and g be a $p^{u(f)}$-th root of f. Let j be such that $\partial_j g \neq 0$. Fix $\rho_0 \in \mathscr{C}(f - a_1, \dots, f - a_m, g, \partial_j g)$ and $r \in]\rho_0, +\infty[$.

By Corollary 4.2.15, there is an index k_r between 1 and m such that

$$(m-1)T(r, g) \leq \sum_{i=1}^{m} Z(r, g - b_i) - Z(r, g - b_{k_r}) + O(1),$$

so, by the first assertion of Lemma 4.2.16, we have

$$(m-1)T(r,g) \leq \sum_{i=1}^{m} \overline{Z}(r, g-b_i) + Z(r, \partial_j g) - Z(r, f-b_{k_r}) - Z_0^S(r, \partial_j g) + O(1).$$

On the other hand, since $\partial_j g \neq 0$, by Lemma 4.2.17, we have

$$Z(r, \partial_j g) \leq Z(r, g-b_{k_r}) + \overline{N}(r,g) - N_0^S(r, \partial_j g) - \log r + O(1),$$

so we have

$$(m-1)T(r,g) \leq \sum_{i=1}^{m} \overline{Z}(r, g-b_i) + \overline{N}(r,g) - Z_0^S(r, \partial_j g) - N_0^S(r, \partial_j g) - \log r + O(1), \quad (4.4)$$

(so in particular, the theorem is proven if $u(f) = 0$).

Moreover, we have (see Definition 4.2.5):

1. $Z(r,f) = Z\left(r, g^{q^{u(f)}}\right) = q^{u(f)} Z(r,g)$,
2. $N(r,f) = N\left(r, g^{q^{u(f)}}\right) = q^{u(f)} N(r,g)$,
3. $\overline{Z}(r, f-a_i) = \overline{Z}(r, g-b_i)$, and finally
4. $\overline{N}(r,f) = \overline{N}(r,g)$.

Combining items 1 and 2 above, we get

$$T(r,f) = q^{u(f)} T(r,g),$$

so that (4.4) becomes:

$$\frac{(m-1)}{q^{u(f)}} T(r,f) \leq \sum_{i=1}^{m} \overline{Z}(r, f-b_i) + \overline{N}(r,f) - Z_0^S(r, \partial_j g) - N_0^S(r, \partial_j g) - \log r + O(1),$$

which was to be proved. □

4.3 AS-BE uniqueness Theorem in several variables

As a direct application of Nevanlinna's Second Fundamental Theorem, we present the AS-BE uniqueness theorem for meromorphic functions in several variables.

Lemma 4.3.1. *If f and g are two non-zero meromorphic functions such that*

$$\operatorname{supp}(f+a)=\operatorname{supp}(g+a)$$

for some $a\in\mathbb{F}$, then

$$\operatorname{supp}(f+a)\subseteq\operatorname{supp}(f-g).$$

Proof. Write $f+a$ and $g+a$ in reduced form as

$$f+a=\frac{f_1}{f_2}\quad\text{and}\quad g+a=\frac{g_1}{g_2}$$

for some non-zero entire functions f_1, f_2, g_1 and g_2. We have

$$f-g=(f+a)-(g+a)=\frac{f_1g_2-f_2g_1}{f_2g_2}.$$

If P is an irreducible divisor of f_1, by assumption, it is an irreducible divisor of g_1, but it is not an irreducible factor of f_2g_2. Thus, P must be in $\operatorname{supp}(f-g)$. □

We now prove Theorem 3, that we restate here for convenience of the reader.

Theorem 4.3.2 (Adams, Straus, Boutabaa, Escassut)**.** *Let a_1, a_2, a_3 and a_4 be four distinct elements of $\mathbb{F}$. If f and g are two meromorphic functions such that $u(f)=u(g)$ and*

$$\operatorname{supp}(f+a_j)=\operatorname{supp}(g+a_j)\tag{4.5}$$

for each $j\in\{1,2,3,4\}$, then $f=g$.

Proof. Consider $m=2$ if f and g are entire, and $m=4$ otherwise. The proof that we present here, for $m=2$, is indeed an alternative proof to Theorem 1.

Without loss of generality, we may assume that $u(f)=0$. Indeed, suppose that the theorem is true in that case and let f,g be such that $s=u(f)\neq 0$. Consider h_1 and h_2 such that

$$f=h_1^{p^s}\quad\text{and}\quad g=h_2^{p^s},$$

and, for each $j=1,\dots,m$, choose a p^s-th root b_j of a_j. We have then

$$\operatorname{supp}(h_1+b_j)=\operatorname{supp}(f+a_j)=\operatorname{supp}(g+a_j)=\operatorname{supp}(h_2+b_j).$$

Hence, since $u(h_1)=0=u(h_2)$, we conclude that $h_1=h_2$, hence $f=g$.

Since by Lemma 4.3.1 we have

$$\operatorname{supp}(f+a_j)\subseteq\operatorname{supp}(f-g)$$

for each $j=1,\dots,m$, we get

$$\sum_{i=1}^{m}\overline{Z}(r,f+a_i)\leq\overline{Z}(r,f-g)+O(1),$$

By Corollary 4.2.9 of the Nevanlinna Second Fundamental Theorem, we obtain

$$\begin{aligned}(m-1)T(r,f) &\leq \sum_{i=1}^{m} \overline{Z}(r,f-a_i) + \overline{N}(r,f) - \log r + O(1)\\ &\leq \overline{Z}(r,f-g) + \overline{N}(r,f) - \log r + O(1),\end{aligned}$$

and then

$$(m-1)T(r,f) \leq T(r,f-g) + \overline{N}(r,f) - \log r + O(1). \tag{4.6}$$

On the other hand, with an analogous argument applied to g, we obtain

$$(m-1)T(r,g) \leq T(r,f-g) + \overline{N}(r,g) - \log r + O(1). \tag{4.7}$$

If f and g are entire functions, the fourth assertion of Proposition 4.2.13 gives

$$T(r,f-g) \leq \max\{T(r,f),T(r,g)\} + O(1),$$

and since

$$\overline{N}(r,f) = \overline{N}(r,g) = 0,$$

from Equations (4.6) and (4.7) , we obtain

$$(m-1)\max\{T(r,f),T(r,g)\} \leq \max\{T(r,f),T(r,g)\} - \log r + O(1),$$

which, for $m=2$, gives a contradiction.

In the case where f and g are not necessarily entire, we note that

$$\max\{\overline{N}(r,f),\overline{N}(r,g)\} \leq \max\{T(r,f),T(r,g)\} + O(1). \tag{4.8}$$

Thus, from Equations (4.6) and (4.7), and by the sixth assertion of Proposition 4.2.13, we obtain

$$\begin{aligned}(m-1)\max\{T(r,f),T(r,g)\} &\leq T(r,f-g) + \max\{T(r,f),T(r,g)\} - \log r + O(1)\\ &\leq T(r,f) + T(r,g) + \max\{T(r,f),T(r,g)\} - \log r + O(1)\\ &\leq 3\max\{T(r,f),T(r,g)\} - \log r + O(1),\end{aligned}$$

which, for $m=4$, gives a contradiction. □

We now prove a couple of corollaries which are analogues of the Erdös-Woods conjecture for meromorphic functions.

Corollary 4.3.3. *Assume that* $\mathbb{F}$ *has characteristic zero. Let*

$$\{a_1,a_2,a_3,a_4\} \quad \textit{and} \quad \{b_1,b_2,b_3,b_4\}$$

be two subsets (not necessarily disjoint) of four distinct elements of $\mathbb{F}$ *and let* ℓ *be a polynomial in* $\mathbb{F}[X_1, \ldots, X_n]$ *of degree one. If* f *and* g *are two meromorphic functions satisfying*

$$\operatorname{supp}(f + a_j) = \operatorname{supp}(g + a_j) \tag{4.9}$$

and

$$\operatorname{supp}(f + \ell + b_j) = \operatorname{supp}(g + \ell + b_j) \tag{4.10}$$

for each $j \in \{1, 2, 3, 4\}$*, then* $f = g$*.*

Proof. By Theorem 4.3.2, if ∇f and ∇g are non-zero, the equality $f = g$ follows from Equation (4.9). If one of the gradients is assumed to be zero, by Equation (4.9) we conclude that the other must be zero. So, since $\nabla(f + \ell) \neq 0$ and $\nabla(g + \ell) \neq 0$, by Equation (4.10) and Theorem 4.3.2, we conclude that $f + \ell = g + \ell$, hence $f = g$. □

Corollary 4.3.4. *Assume that* $\mathbb{F}$ *has characteristic* $p > 3$ *and let* ℓ *be a polynomial in* $\mathbb{F}[X_1, \ldots, X_n]$ *of degree one. If* f *and* g *are two non-constant meromorphic functions satisfying*

$$\operatorname{supp}(f + a) = \operatorname{supp}(g + a) \tag{4.11}$$

and

$$\operatorname{supp}(f + \ell + a) = \operatorname{supp}(g + \ell + a) \tag{4.12}$$

for each $a \in \{0, 1, 2, 3\} \cap \mathbb{F}$*, then* $f = g$*.*

Proof. Similarly to the characteristic zero case, if ∇f and ∇g are both non-zero or both zero, the equality $f = g$ follows from Theorem 4.3.2 and Equations (4.11) and (4.12). We claim that there is no other case.

Assume, for instance, that $\nabla f \neq 0$ and $\nabla g = 0$. In particular, this implies that $f \neq g$. Moreover, Corollary 4 implies that $g = f^{p^m}$, for some positive integer m.

We prove that $\nabla(f + \ell) = 0$. Assume the contrary. By Equation (4.12), we have $\operatorname{supp}(f+\ell) = \operatorname{supp}(g+\ell)$, hence $\nabla(g+\ell) \neq 0$, so by Theorem 4.3.2, we have $f+\ell = g+\ell$, hence $f = g$, which is a contradiction.

Observe that since $\nabla(g) = 0$, we have $\nabla(g+\ell) \neq 0$. Since $\nabla(f+\ell) = 0$, by Corollary 4, there is a positive integer k such that

$$g + \ell = (f + \ell)^{p^k},$$

so that we have

$$\ell = f^{p^m} - (f + \ell)^{p^k} = \left(f^{p^{m-1}} - (f + \ell)^{p^{k-1}}\right)^p,$$

which is impossible. □

4.4 Branched values of meromorphic functions

This section is dedicated to prove Theorem 9.

Consider a non-zero entire function f whose decomposition as a product of irreducible factors is

$$f = \prod_{j \in J} P_j^{d_j}.$$

We will say that an entire function g is a *square free part* of f if g is associated to the entire function

$$\prod_{\substack{j \in J \\ d_j = 1}} P_j,$$

namely, if up to a constant, g is the product of the simple irreducible factors of f.

Definition 4.4.1. Given a non-zero meromorphic function f, we say that $b \in \mathbb{F}$ is a *perfectly branched value* for f, if the square-free part of the function of zeros of $f - b$ is a polynomial. Analogously, we say that b is a *totally branched value* for f if the square-free part of the function of zeros of $f - b$ is a unit.

Remark 4.4.2. The following are some trivial situations:

1. If a meromorphic function f is in the field of rational functions $\mathbb{F}(X_1, \ldots, X_n)$, each $b \in \mathbb{F}$ is a perfectly branched value for f.

2. When the ground field $\mathbb{F}$ has characteristic $p > 0$, for each meromorphic function f which is a p-th power, each $b \in \mathbb{F}$ is a totally branched value for f.

We now prove a few lemmas in order to prove items 1, 2 and 4 of Theorem 9 - this will be Theorems 4.4.6, 4.4.7 and 4.4.8 below.

Lemma 4.4.3. *A non-zero meromorphic function f is transcendental if and only if it satisfies*

$$\lim_{r \to \infty} \frac{T(r, f)}{\log r} = +\infty$$

Proof. This is an immediate consequence of Corollary 2.5.2. □

Lemma 4.4.4. *Let f be a non-zero meromorphic function. If b is a perfectly branched value for f, then there is a constant $M \geq 0$ such that*

$$\overline{Z}(r, f - b) \leq \frac{1}{2} T(r, f) + M \log r + O(1).$$

Moreover, if b is a totally branched value for f, then one can take $M = 0$.

Proof. Let $f_0 \in \mathcal{E}_n$ be a function of zeros of $f - b$. Write $f_0 = gh$ where g is the square-free part of f_0. Since

$$\mathcal{D}(g) \geq \overline{\mathcal{D}}(g) \quad \text{and} \quad \mathcal{D}(h) \geq 2\overline{\mathcal{D}}(h)$$

(see Definition 4.2.5 for the notation $\overline{\mathcal{D}}(\cdot)$), we obtain

$$\begin{aligned} 2\overline{\mathcal{D}}(f_0) &= 2\overline{\mathcal{D}}(g) + 2\overline{\mathcal{D}}(h) \\ &\leq \mathcal{D}(g) + (\mathcal{D}(g) + \mathcal{D}(h)) \\ &\leq \mathcal{D}(g) + \mathcal{D}(f_0). \end{aligned}$$

By Lemma 4.1.3, it follows that

$$\overline{Z}(r, f - b) \leq \frac{1}{2}\left(Z(r, f - b) + Z(r, g)\right).$$

Since g is a polynomial, we have

$$Z(r, g) = \deg g \log(r) + O(1).$$

Thus we have

$$\begin{aligned} \overline{Z}(r, f - b) &\leq \frac{1}{2}\left(Z(r, f - b) + \deg g \log r\right) + O(1) \\ &\leq \frac{1}{2}T(r, f) + M \log r + O(1), \end{aligned}$$

where

$$M = \frac{1}{2}\deg g.$$

If b is a non-zero totally branched value for f, then

$$M = \frac{1}{2}\deg g = 0.$$

When $b = 0$, we have

$$2\overline{\mathcal{D}}(f_0) \leq \mathcal{D}(f_0),$$

from what we obtain

$$\overline{Z}(r, f - b) \leq \frac{1}{2}T(r, f) + O(1),$$

so we can take $M = 0$. □

Lemma 4.4.5. *Let f be a non-constant meromorphic function satisfying $u(f) = 0$. If f admits m distinct perfectly branched values, then there is a constant M such that*

$$\frac{(m-2)}{2}T(r, f) \leq \overline{N}(r, f) + (M - 1)\log r + O(1).$$

Moreover, if each of the m values are totally branched values for f, then one can take $M = 0$.

Proof. Let $\{b_1, \ldots, b_m\}$ be a finite set of m distinct perfectly branched values for f. By Lemma 4.4.4, for each $j = 1, \ldots, m$ there is a constant $M_j \geq 0$ such that

$$\overline{Z}(r, f - b_j) \leq \frac{1}{2}T(r, f) + M_j \log r + O(1).$$

Thus, by Corollary 4.2.9 of the Nevanlinna's Second Fundamental Theorem we obtain

$$\begin{aligned}(m-1)T(r, f) &\leq \sum_{j=1}^{m} \overline{Z}(r, f - b_j) + \overline{N}(r, f) - \log r + O(1) \\ &\leq \frac{m}{2}T(r, f) + \overline{N}(r, f) + (M-1)\log r + O(1),\end{aligned}$$

where

$$M = \sum_{j=1}^{m} M_j.$$

Moreover, Lemma 4.4.4 states that if b_j is a totally branched value for f, then $M_j = 0$, which concludes the proof. □

We can now prove items 1, 2, and 4 of Theorem 9.

Theorem 4.4.6. *A transcendental meromorphic function f satisfying $u(f) = 0$ has at most four perfectly branched values.*

Proof. Let f be a transcendental meromorphic function which admits m distinct perfectly branched values. By Lemma 4.4.5, there exists a constant M such that

$$\begin{aligned}\frac{(m-2)}{2}T(r, f) &\leq \overline{N}(r, f) + (M-1)\log r + O(1) \\ &\leq T(r, f) + (M-1)\log r + O(1).\end{aligned}$$

Thus, we obtain the relation

$$(m-4)\frac{T(r, f)}{\log r} \leq O(1),$$

which contradicts Lemma 4.4.3 when $m > 4$. □

Theorem 4.4.7. *A non-constant meromorphic function f satisfying $u(f) = 0$ has at most three totally branched values.*

Proof. Let m be the number of distinct totally branched values for f. Lemma 4.4.5 implies that

$$\frac{(m-2)}{2}T(r, f) \leq T(r, f) - \log r + O(1),$$

from which we deduce

$$\frac{(m-4)}{2}T(r,f) \leq -\log r + O(1). \tag{4.13}$$

If f is transcendental, then from Theorem 4.4.6 we have $m \leq 4$, and the latter inequality gives a contradiction for $m = 4$.

If f is a rational function, then it satisfies

$$T(r,f) = \deg(f)\log r + O(1),$$

where $\deg(f)$ is the maximum between $\deg(f_0)$ and $\deg(f_\infty)$. If $\deg(f) = 1$, then f has no totally branched values. If $\deg f \geq 2$, then from (4.13) and the latter equality, we obtain

$$(m-3)\log r \leq O(1),$$

which gives $m \leq 3$. □

Theorem 4.4.8. *A non-constant polynomial f satisfying $u(f) = 0$ has at most one totally branched value.*

Proof. First note that the statement is trivial if f has degree 1. Assume that f has degree at least 2. Let m be the number of distinct totally branched values. By Lemma 4.4.5, we have

$$\frac{(m-2)}{2}T(r,f) \leq -\log r + O(1).$$

On the other hand, we have

$$T(r,f) = \deg f \log r + O(1).$$

Since $\deg(f) \geq 2$, it follows that

$$(m-2)\log r \leq -\log r + O(1),$$

which implies $m \leq 1$. □

Before we can prove item 3 of Theorem 9 (this is Theorem 4.4.10 below), we need one more lemma (which will also be used in the next section).

The proof of the next result requires to extend the maximum term to elements of an algebraic closure of $K = \mathcal{M}_n$. In the first chapter, Proposition 1.1.8 gives a precise way to extend each ultrametric absolute value $|\ |_r$ of K to an ultrametric absolute value on its algebraic closure $\overline{K}$. Given $\Theta \in \overline{K}$, for each $r \in \mathbb{R}_{\geq 0}$, we define

$$|\Theta|(r) = \sqrt[d]{|P_\Theta(0)|(r)},$$

where $P_\Theta \in K[t]$ denotes the minimal polynomial of Θ over K, and d denotes the degree of P_Θ. As for meromorphic functions, we will say that Θ is *transcendental* if

$$\lim_{r\to\infty} \frac{|\Theta|(r)}{r^k} = +\infty$$

for each $k \in \mathbb{N}$. It is important to notice that elements of $\overline{K} \setminus K$ are purely formal objects with no meaning as global meromorphic functions.

Lemma 4.4.9. *Consider two entire functions g and h whose quotient g/h is transcendental. Let $d \in \mathbb{Z}_{\geq 2}$ be coprime with q. Given a non-zero rational function F, the meromorphic function $g^d - Fh^d$ satisfies*

$$\lim_{r\to\infty} \frac{|g^d - Fh^d|(r)}{r^k} = +\infty$$

for each $k \in \mathbb{N}$.

Proof. First, we notice that $g^d - Fh^d$ is non-zero, because g/h is transcendental. Assume that h is transcendental (otherwise the assertion will follow from Lemma 2.5.4 applied to g). Let $\Theta \in \overline{K}$ be a d-th root of F. Thus $g^d - Fh^d$ can be written as

$$g^d - Fh^d = (g - \Theta h)S,$$

where $\Xi = g - \Theta h$ and

$$S = \sum_{i=0}^{d-1} g^{d-1-i}(g - \Xi)^i.$$

It is enough to prove that if Ξ is not transcendental, then S is. So assume that Ξ is not transcendental.

Note that S can be written as

$$dg^{d-1} - H,$$

where H is a polynomial in Ξ of degree $d-1$ whose coefficients are polynomials in g of degree at most $d-2$. Since Ξ is not transcendental, there exists $a \in |\mathbb{F}^*|$ and $\ell \in \mathbb{N}$ such that for each $r \in \mathbb{R}_{\geq 0}$ we have

$$|\Xi|(r) \leq ar^{\ell/d},$$

hence

$$|H|(r) \leq a^{d-1} r^{\frac{\ell(d-1)}{d}} (|g|(r))^{d-2}.$$

On the other hand, since g is transcendental and because d is coprime with q by hypothesis, by Lemma 2.5.4, we have for r large enough

$$|d|(|g|(r))^{d-1} > a^{d-1} r^{\frac{\ell(d-1)}{d}} (|g|(r))^{d-2},$$

hence

$$|H|(r) < |d|(|g|(r))^{d-1},$$

hence

$$|dg^{d-1} - H|(r) = |d|(|g|(r))^{d-1}.$$

Therefore, S is transcendental. □

Theorem 4.4.10. *A transcendental meromorphic function f satisfying $u(f) = 0$, and whose function of poles f_∞ is polynomial, has at most one perfectly branched value.*

Proof. Assume that f has two distinct perfectly branched values b_1 and b_2. Choose $d = 3$ if $\mathbb{F}$ has characteristic 2, and $d = 2$ otherwise.

Suppose first that for each $j \in \{1, 2\}$, we have

$$f - b_1 = \frac{Pg^d}{f_\infty} \qquad \text{and} \qquad f - b_2 = \frac{Qh^d}{f_\infty}, \tag{4.14}$$

for some polynomials P and Q, and transcendental entire functions g and h. This means that the irreducible factors of the non square-free part of the function of zeros of $f - b_j$, whose multiplicity is distinct from d, is a finite subset of $\mathbb{F}[X_1, \ldots, X_n]$.

We prove that g/h is transcendental. Assume it is not. By hypothesis, the fraction

$$\frac{f_0 - b_1 f_\infty}{f_0 - b_2 f_\infty}$$

is a rational function. Since each $f_0 - b_i f_\infty$ is transcendental by hypothesis, the greatest common divisor η of $f_0 - b_1 f_\infty$ and $f_0 - b_2 f_\infty$ is a transcendental entire function. But then η divides the polynomial

$$f_0 - b_1 f_\infty - (f_0 - b_2 f_\infty),$$

which is absurd.

Writing

$$F = \frac{Q}{P} \quad \text{and} \quad G = (b_2 - b_1)\frac{f_\infty}{P},$$

we have

$$g^d - Fh^d = G.$$

Since f_∞ is a polynomial by hypothesis, G is a rational function, hence there exists $\ell \in \mathbb{N}$ such that

$$\frac{|G|(r)}{r^\ell}$$

is bounded (by Lemma 4.4.3). But at the same time, since $\frac{g}{h}$ is transcendental, the meromorphic function $g^d - Fh^d$ satisfies the assertion of Lemma 4.4.9, hence is transcendental, which is absurd. Therefore, not both $f - b_1$ and $f - b_2$ can be written as in Equations (4.14). Assume that $f - b_1$ cannot be written in that form.

From the previous discussion, we deduce that the set of irreducible factors of the zero function of $f-b_1$, whose order of multiplicity is distinct from d, is either an infinite subset of $\mathbb{F}[X_1,\dots,X_n]$ (case 1), or it contains at least one transcendental irreducible entire function (case 2).

We now prove that in both cases the function

$$\sigma(r) = Z(r, f-b_1) - 2\overline{Z}(r, f-b_1)$$

satisfies

$$\lim_{r\to\infty} \frac{\sigma(r)}{\log r} = +\infty.$$

Write $f-b_1 = f_1/f_\infty$ in reduced form. The divisor $\mathcal{D}(f_1)$ can be written in the form

$$\mathcal{D}(f_1) = D_0 + 2D_1 + \sum_i d_i \mathcal{P}_i,$$

where D_0 is the divisor of the square-free part, D_1 corresponds to the points with multiplicity exactly 2, and each d_i is at least 3. Assume that the characteristic p is not 2. We have

$$\overline{\mathcal{D}}(f_1) = D_0 + D_1 + \sum_i e_i \mathcal{P}_i,$$

where $e_i = 1$ if d_i is coprime with p and $e_i = 0$ otherwise. Since $d = 2$, we have

$$\mathcal{D}(f_1) + D_0 = 2\overline{\mathcal{D}}(f_1) + \sum_i (d_i - 2e_i)\mathcal{P}_i,$$

hence

$$Z(r, f_1) + Z(r, D_0) = 2\overline{Z}(r, f_1) + \sum_i (d_i - 2e_i) Z(r, \mathcal{P}_i).$$

We deduce that

$$\begin{aligned}
\frac{\sigma(r)}{\log r} &= \frac{Z(r, f_1)}{\log r} - \frac{2\overline{Z}(r, f_1)}{\log r} \\
&= \frac{\sum_i (d_i - 2e_i) Z(r, \mathcal{P}_i)}{\log r} - \frac{Z(r, D_0)}{\log r} \\
&= \frac{\sum_i (d_i - 2e_i) Z(r, \mathcal{P}_i)}{\log r} - O(1),
\end{aligned}$$

which proves our claim since each d_i is at least 3. If $p = 2$, we have

$$\overline{\mathcal{D}}(f_1) = D_0 + \sum_i e_i \mathcal{P}_i,$$

so that

$$\begin{aligned}
\mathcal{D}(f_1) - 2\overline{\mathcal{D}}(f_1) &= D_0 + 2D_1 + \sum_i d_i \mathcal{P}_i - 2(D_0 + \sum_i e_i \mathcal{P}_i) \\
&= -D_0 + 2D_1 + \sum_i (d_i - 2e_i)\mathcal{P}_i.
\end{aligned}$$

We leave to the reader the simple verification that the counting function of zeros of the divisor

$$2D_1 + \sum_i (d_i - 2e_i)\mathcal{P}_i$$

has order of growth greater than that of $\log r$ in both case 1 and case 2.

On the other hand, from the relations

$$\begin{aligned}\overline{Z}(r, f - b_1) &= \frac{Z(f - b_1) - \big(Z(r, f - b_1) - 2\overline{Z}(r, f - b_1)\big)}{2} \\ &\leq \frac{T(r, f - b_1) - \sigma(r)}{2} \\ &= \frac{T(r, f) - \sigma(r)}{2} + O(1),\end{aligned}$$

combined with Lemma 4.4.4 applied to $f - b_2$, and by Corollary 4.2.9 of the Nevanlinna's Second Fundamental Theorem, we obtain

$$(m-1)T(r, f) \leq \frac{mT(r, f) - \sigma(r)}{2} + O(\log r),$$

with $m = 2$, which gives a contradiction. □

4.5 Applications of branched values to solve some functional equations

We first prove Corollary 10, stated in the introduction.

Proof of Corollary 10. Consider the equation

$$Pf^d - Qg^m = R, \tag{4.15}$$

in the variables f and g, where P, Q and R are three non-zero polynomials, and where $d, m \in \mathbb{Z}_{\geq 2}$ are coprime with the characteristic exponent q. We want to prove that there are no transcendental solutions.

For the sake of contradiction, assume that f and g are transcendental solutions. Consider the meromorphic functions

$$F = \frac{Pf^d}{R} \quad \text{and} \quad G = \frac{Qg^m}{R}.$$

Note that both are transcendental and that their respective functions of poles are polynomials. Notice that since $F - G = 1$, we have $u(F) = u(G)$. Since 0 and 1 are perfectly branched values for F, it follows that $u(F)$ must be positive (and consequently,

p is positive), otherwise it would contradict Theorem 4.4.10. Set $\ell = u(F)$ and let $F_0, G_0 \in \mathcal{M}_n$ be such that

$$F = F_0^{p^\ell} \quad \text{and} \quad G = G_0^{p^\ell},$$

so that we have $F_0 - G_0 = 1$, and their functions of poles are polynomials.

Note that in reduced form, F can be written as

$$F = \frac{P' f'^d}{R'}$$

where P' and R' are polynomials, and f' is a transcendental function which is coprime with P'. Since F is a p^ℓ-th power, its function of zeros $P' f'^d$ is a p^ℓ-th power. Since moreover P' is coprime with f', both P' and f'^d are p^ℓ-th powers, and since d is coprime with p, indeed f' is a p^ℓ-th power. So actually, the function of zeros of F can be written as

$$(P_0 f_0^d)^{p^\ell}$$

where P_0 is a polynomial, and f_0 is a transcendental entire function which is coprime with P_0. Similarly, the function of zeros of G can be written as

$$(Q_0 g_0^m)^{p^\ell},$$

where Q_0 is a polynomial, and g_0 is a transcendental entire function which is coprime with g_0. Therefore, $P_0 f_0^d$ and $Q_0 g_0^m$ are the functions of zeros of F_0 and G_0, respectively. Thus, we obtain that 0 and 1 are perfectly branched values for F_0 (since $F_0 - 1 = G_0$ and 0 is a perfectly branched values for G_0). This contradicts Theorem 4.4.10. □

Next we prove Theorem 11, stated in the introduction.

Proof of Theorem 11. Consider the equation

$$g^m = h f^m + Q \tag{4.16}$$

in the variables f and g, where $m \geq 2$ is an integer coprime with the characteristic exponent q, $h \in \mathcal{E}_n$ is an non-zero m-th power, and $Q \in \mathbb{F}[X_1, \ldots, X_n]$ is non-zero. We want to prove that none of f and g can be transcendental, and that if h is transcendental then the equation has no solution, namely, we want to prove that if the equation admits a solution (f, g), then none of f, g and h can be transcendental.

If h is a polynomial, then Corollary 10 implies that f and g must be polynomials. So we need only to prove that h is not transcendental. Assume it is. Since h is an m-th power, it can be written as $h = H^m$ for some transcendental entire function H, so that Equation (4.16) becomes

$$g^m = H^m f^m + Q.$$

If the meromorphic function Hf/g is a polynomial, then the entire function

$$G = \gcd(Hf, g)$$

is transcendental, as H is, and this contradicts the fact that G divides Q. Hence Hf/g is transcendental.

Since m is coprime with q, by Lemma 4.4.9, we obtain that $g^m - (Hf)^m = Q$ is transcendental, which is absurd. □

Finally, we prove Corollary 14.

Proof of Corollary 14. Let f and g be two coprime entire functions such that

$$\nabla\left(\frac{f}{g}\right) \neq 0.$$

Assume that $\{\lambda_1, \dots, \lambda_N\}$ is a subset of $\mathbb{F}$ such that $f + \lambda_i g$ is a power in $\mathcal{E}_n$, for each $i = 1, \dots, N$. For each $i = 1, \dots, N$, write

$$f + \lambda_i g = h_i^{m_i}$$

for some $h_i \in \mathcal{E}_n$ and $m_i \in \mathbb{Z}_{\geq 2}$. The coprimality between f and g implies the coprimality between g and h_i for each i. By setting $F = f/g$, we obtain

$$F + \lambda_i = \frac{h_i^{m_i}}{g},$$

namely, for each $i \in \{1, \dots, N\}$, $-\lambda_i$ is a totally branched value for F. Since $\nabla(F) \neq 0$, we have $u(F) = 0$ by Corollary 2.7.7. Thus, by Theorem 4.4.7, it follows that N is at most 3. □

4.6 Another application of branched values for analytic functions in the stripped balls

This section is dedicated to prove Theorems 12 and 13. Here, we assume that $\mathbb{F}$ has characteristic zero. We will use κ to denote the residual characteristic of $\mathbb{F}$ (namely, the characteristic of the residual field of $\mathbb{F}$). We will fix $a \in \mathbb{F}$ and $\rho^- \in \mathbb{R}_{\geq}$. For a general introduction to non-Archimedean analysis in one variable, we refer the reader to [Es1].

We first state a few theorems and lemmas that are needed for the proofs.

Theorem 4.6.1. *[Es1, Cor 23.17] An analytic function h in the disk $d(a, \rho^-)$ is a unit if and only if, for each $x \in d(a, \rho^-)$, it satisfies*

$$|h(x) - h(a)| < |h(a)|.$$

The following lemma is well known.

Lemma 4.6.2. *Let $f, g \in \mathcal{A}(d(a,\rho^-))$. If the product fg is bounded, then f and g are both bounded.*

Theorem 4.6.3. *[Es1, Ch. 33] Let $m \in \mathbb{Z}_{\geq 2}$ and $w \in \mathcal{A}(d(a,\rho^-))$ be such that*

- *for each $x \in d(a,\rho^-)$, the analytic function w satisfies*

$$|w(x) - 1| < 1;$$

if $\kappa \neq 2$ and m is coprime with κ; or

- *for each $x \in d(a,\rho^-)$, the analytic function w satisfies*

$$|w(x) - 1| < \frac{1}{2},$$

if $\kappa = m = 2$.

In both situations, w has an m-th root in $\mathcal{A}(d(a,\rho^-))$.

Theorem 4.6.4. *[La, Th. 2.3.7] Suppose that $\mathbb{F}$ is spherically complete. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements of $d(a,\rho^-)$ satisfying $\lim\limits_{n\to\infty} |a_n| = \rho$ and let $(d_n)_{n\in\mathbb{N}}$ be any sequence of positive integers. There exists $f \in \mathcal{A}(d(a,\rho^-))$ admitting each a_n as a zero of order d_n.*

The following lemma is proven in the same way as Lemma 4.4.9 (and the proof is indeed simpler).

Lemma 4.6.5. *Let $f, g \in \mathcal{A}_u(d(0,\rho^-))$ with $f \neq g$. For each $m \in \mathbb{Z}_{\geq 2}$, we have $f^m - g^m \in \mathcal{A}_u(d(a,\rho))$.*

Lemma 4.6.6. *Let $m \in \mathbb{Z}_{\geq 2}$ and let $\mathbb{L}$ be a complete algebraically closed extension of $\mathbb{F}$. Let*

$$d_{\mathbb{L}}(a,\rho^-) = \{x \in \mathbb{L} \colon |x - a| \leq \rho^-\}.$$

If f has an m-th root g in the ring $\mathcal{A}(d_{\mathbb{L}}(0,\rho^-))$, then $g \in \mathcal{A}(d(a,\rho^-))$.

Proof. Without loss of generality, we assume $a = 0$. Write

$$f = \sum_{i=0}^{\infty} a_i x^i \quad \text{and} \quad g = \sum_{i=0}^{\infty} b_i x^i,$$

where $a_i \in \mathbb{F}$ and $b_i \in \mathbb{L}$. By induction, we prove that $b_i \in \mathbb{F}$ for each $i \in \mathbb{N}$. Since $\mathbb{F}$ is algebraically closed and $a_0^m = b_0$, we obtain $b_0 \in \mathbb{F}$. Now suppose that we have proven that $a_j \in \mathbb{F}$ for each $0 \leq j \leq k-1$. It is clear that b_k has the form $a_k a_0^{m-k} + v$ where v is a polynomial in a_0, a_1, ..., a_{k-1}. Therefore, it also belongs to $\mathbb{F}$. Consequently, each coefficient of g is in $\mathbb{F}$. □

If f is an analytic function in a stripped disk, by $\sqrt[m]{f}$, we mean an arbitrary m-th root of f.

Theorem 4.6.7. *Let $m \in \mathbb{Z}_{\geq 2}$ be coprime with κ and let $f \in \mathcal{A}(d(a, \rho^-))$. If all zeros of f have order a multiple of m, then f has an m-th root $g \in \mathcal{A}(d(a, \rho^-))$.*

Proof. Suppose first that $\mathbb{F}$ is spherically complete. Let $(\alpha_n)_{n\in\mathbb{N}}$ be the sequence of zeros of the function h in $d(a, \rho^-)$, with $\lim\limits_{n\to+\infty} |\alpha_n| = \rho$, each α_n having order ms_n.

By Theorem 4.6.4, there exists $\phi \in \mathcal{A}(d(a, \rho^-))$ whose zeros are given by the sequence $(\alpha_n)_{n\in\mathbb{N}}$ and whose respective order of multiplicities are s_n. Thus, $\frac{f}{\phi^m}$ has no zero and no pole and therefore is an invertible bounded function ℓ which belongs to $\mathcal{A}_b(d(a, \rho^-))$. Let $\psi = \frac{\ell}{\ell(a)}$ and let λ be an m-th root of $\ell(a)$. Since ℓ has no zero in $d(a, \rho^-)$, for each $x \in d(a, \rho^-)$, we have

$$|\ell(x) - \ell(a)| < |\ell(a)|,$$

and therefore

$$|\psi(x) - \psi(a)| = |\psi(x) - 1| < 1.$$

Consequently, since κ does not divide m, by Theorem 4.6.3, the function $\sqrt[m]{\psi(x)}$ belongs to $\mathcal{A}_b(d(a, \rho^-))$. So, we have

$$f(x) = \left(\lambda\phi(x)\sqrt[m]{\psi(x)}\right)^m,$$

which proves the claim when $\mathbb{F}$ is spherically complete.

Consider now the general case, when $\mathbb{F}$ is no longer supposed to be spherically complete. Let $\widehat{\mathbb{F}}$ be a spherically complete algebraically closed extension of $\mathbb{F}$. Given a disk $d(a, \rho^-)$ of $\mathbb{F}$, we will denote by $\widehat{d}(a, \rho^-)$ the similar disk of $\widehat{\mathbb{F}}$:

$$\{x \in \widehat{\mathbb{F}} \mid |x - a| < r\}.$$

The function f can be extended to a function $\widehat{f}$ which belongs to $\mathcal{A}(\widehat{d}(\alpha, \rho^-))$, and hence there exists a function $g \in \mathcal{A}(\widehat{d}(\alpha, \rho^-))$ such that $g^m = f$. By Lemma 4.6.6, g is a power series that has all its coefficients in $\mathbb{F}$, and hence belongs to $\mathcal{A}((d(a, \rho^-))$. □

We are now able to prove Theorem 12.

Proof of Theorem 12. Let $m \in \mathbb{Z}_{\geq 2}$. Let $h \in \mathcal{A}(d(a, \rho^-))$ and $w \in \mathcal{A}_b(d(a, \rho^-))$ be non-zero. We consider the functional equation

$$g^m = hf^m + w \tag{4.17}$$

in the variables f and g. Suppose that Equation (4.17) has solutions $f,\ g \in \mathcal{A}_u(d(a, R^-))$. We assume that one of the following hypothesis is satisfied.

- m is coprime with the residual characteristic κ of $\mathbb{F}$ and h has no zeros in $d(a, \rho^-)$.
- m and the residual characteristic of $\mathbb{F}$ are equal to 2, and for each $x \in d(a, \rho^-)$ we have
$$|h(x) - h(0)| < \frac{1}{2}|h(0)|.$$
- m is coprime with the residual characteristic of $\mathbb{F}$ and each zero of h has order of multiplicity divisible by m.

By Theorem 4.6.3 and Lemma 4.6.6, there exists a function $\phi \in \mathcal{A}(d(a, R^-))$ such that $h = \phi^m$. Consequently we can write
$$(g(x))^m - (\phi(x)f(x))^q = w(x).$$
But by Lemma 4.6.5,
$$(g(x))^m - (\phi(x)f(x))^q$$
is unbounded, a contradiction to the hypothesis on w. □

In order to prove Theorem 13 we need to recall the following definitions. Given $f, w \in \mathcal{M}(d(0, \rho^-))$, the function w is called a *small function* with respect to f if it satisfies
$$\lim_{r \to \rho} \frac{T(r, w)}{T(r, f)} = 0.$$
We denote by $\mathcal{M}_f(d(0, \rho^-))$ the set of functions $w \in \mathcal{M}(d(0, \rho^-)$ which are small functions with respect to f. Similarly, we denote by $\mathcal{A}_f(d(0, \rho^-))$, the set of functions $w \in \mathcal{A}_f(d(0, \rho^-))$ which are small functions respect to f. Note that given any $f \in \mathcal{M}_u(d(0, R^-))$, all functions $u \in \mathcal{M}_b(d(0, \rho^-))$ belong to $\mathcal{M}_f(d(0, \rho^-))$. In the proof of Theorem 13, we will use the following Lemma 4.6.8 and Theorem 4.6.10, known as Nevanlinna's Second Fundamental Theorem on 3 small functions.

Lemma 4.6.8. *Given $f \in \mathcal{M}(d(0, \rho^-))$, the set $\mathcal{M}_f(d(0, \rho^-))$ is a subfield of $\mathcal{M}(d(0, \rho^-))$.*

Lemma 4.6.9. *A function $f \in \mathcal{A}(d(0, \rho^-))$ belongs to $\mathcal{A}_b(d(0, \rho^-))$ if and only if $T(r, f)$ is bounded when r tends to ρ.*

Theorem 4.6.10. *[HuYa, Th. 2.21] Let $f \in \mathcal{M}_u(d(0, \rho^-))$ and $u_1, u_2 \in \mathcal{M}_f(d(0, \rho^-))$ be distinct. We have*
$$T(r, f) \leq \overline{Z}(r, f - u_1) + \overline{Z}(r, f - u_2) + \overline{N}(r, f) + o(T(r, f)).$$

Corollary 4.6.11. *Let $f \in \mathcal{A}_u(d(0, \rho^-))$ and $u \in \mathcal{A}_f(d(0, \rho^-))$. We have*
$$T(r, f) \leq \overline{Z}(r, f) + \overline{Z}(r, f - u) + o(T(r, f))$$

We can now prove Theorem 13.

Proof of Theorem 13. Let $n, m \in \mathbb{Z}_{\geq 2}$ be such that $\max\{n, m\} \geq 3$. Let $h, w \in \mathcal{A}_b(d(a, \rho^-))$ be non-zero and consider the functional equation

$$g^n = hf^m + w$$

in the variables f and g. We want to prove that it has no solution over $\mathcal{A}_u(d(a, \rho^-))$.

Write $F(x) = g(x)^n$. From Corollary 4.6.11, we have

$$T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, F - w) + o(T(r, F)).$$

On the other hand, we have

$$\overline{Z}(r, F) \leq \frac{1}{n} Z(r, F).$$

Moreover, since h is bounded, by Lemma 4.6.9, $Z(r, h)$ is bounded, hence by Lemma 4.6.8, we have

$$\overline{Z}(r, hf^m) \leq Z(r, f) + Z(r, h) = Z(r, f) + O(1).$$

Therefore, we have

$$\overline{Z}(r, hf^m) \leq \frac{1}{m} Z(r, hf^m) + O(1) = \frac{1}{m} Z(r, F) + O(1).$$

On the other hand, we have

$$Z(r, F) = Z(r, F - w) + O(1) = T(r, F) + O(1),$$

hence

$$T(r, F) \leq \left(\frac{1}{m} + \frac{1}{n}\right) T(r, F) + o(T(r, F)),$$

which implies

$$\frac{1}{m} + \frac{1}{n} \geq 1.$$

The latter contradicts the fact that our hypothesis on m and n implies

$$\frac{1}{m} + \frac{1}{n} \leq \frac{5}{6}.$$

□

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