Universidad de Concepción
Dirección de Postgrado
Facultad de Ciencias Físicas y Matemáticas - Programa Magíster en Matemática

Sistemas Lineales y Topología de $\mathbb{T}$-Variedades de Complejidad uno

(Linear Systems and Topology of Complexity-one $\mathbb{T}$-Varieties)

Tesis para optar al grado de Magíster en Matemática

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Introduction

In this thesis we work in two important problems of Algebraic Geometry: special linear systems through multiple points on $(\mathbb{P}^1)^n$ and the topology of varieties with torus action.

Problem 1 is the content of Chapter 2. There we discuss the problem of determining the dimension of a linear system of hypersurfaces of a projective variety $X \subseteq \mathbb{P}^N$ passing through finitely many points in very general position with prescribed multiplicities. This problem is related to polynomial interpolation in several variables and to the classification of defective higher secant varieties of X. In case $X = \mathbb{P}^2$ the Segre-Harbourne-Gimigliano-Hirschowitz conjecture [**24**, **25**, **28**, **41**] predicts the dimension of such linear systems. Several cases of this conjecture have been proved, see e.g. [**12**, **17**–**19**, **33**, **38**]. In [**36**, **37**] an analogous conjecture is stated for $X = \mathbb{P}^3$ and proved when the multiplicities of the points are ≤ 5 in [**6**, **7**]. There is no such a conjecture for higher dimensional projective spaces, but all the same there are partial results about the dimension of such linear systems [**21**, **42**] and in [**8**] the authors determine the contribution to the dimension of linear systems given by linear subspaces.

Inspired by [**8**] in this thesis we study linear systems of $X = (\mathbb{P}^1)^n$ through multiple points for $n \geq 2$. Let $\mathcal{L}$ be the linear system of hypersurfaces of degree $(d_1, \ldots, d_n)$ in $(\mathbb{P}^1)^n$ passing through a general union of r points with multiplicities respectively $m_1, \ldots m_r$. We denote such linear system by $\mathcal{L}_{(d_1,\ldots,d_n)}(m_1, \ldots, m_r)$. One introduce the *expected dimension* $\operatorname{edim}(\mathcal{L})$ of the linear system (see 2.1.10) which satisfies the inequality $\dim(\mathcal{L}) \geq \operatorname{edim}(\mathcal{L})$ and the equality holds if the conditions imposed by the points are independent. The system is *special* if the inequality is strict and *non-special* otherwise. Special linear systems have been classified when all the multiplicities are ≤ 2 in [**15**, **32**, **34**, **46**].

We prove that a fiber of a projection map $(\mathbb{P}^1)^n \to (\mathbb{P}^1)^s$, where $1 \leq s \leq n-1$, through a multiple point can contribute to the speciality of the linear system $\mathcal{L}$. We introduce in Definition 2.1.10 the *fiber-expected dimension* $\operatorname{efdim}(\mathcal{L})$. This definition takes into account the poossible speciality of $\mathcal{L}$ coming from the fibers of such projections. We say that $\mathcal{L}$ is *fiber special* if the inequality $\dim(\mathcal{L}) > \operatorname{fdim}(\mathcal{L})$ holds and that it is *fiber non-special* otherwise. Our first result is the following.

THEOREM 1. *Given a linear system $\mathcal{L}$ in $(\mathbb{P}^1)^n$, the inequalities* $\dim(\mathcal{L}) \geq \operatorname{efdim}(\mathcal{L}) \geq \operatorname{edim}(\mathcal{L})$ *hold. Moreover, a linear system through two multiple points in $(\mathbb{P}^1)^n$ is fiber non-special.*

If there are more than two multiple points then there are examples of fiber special systems (see Example 2.2.3). To study such linear systems we make use of a degeneration of $(\mathbb{P}^1)^n$ into two copies of $(\mathbb{P}^1)^n$ introduced in [**34**]. We relate the speciality of a linear system with that of the two linear systems arising from the degeneration in the following way.

THEOREM 2. *Let $\mathcal{L}_{(d_1,\ldots,d_n)}(m_1, \ldots, m_r)$ a linear system in $(\mathbb{P}^1)$. And consider the linear systems $\mathcal{L}_1 = \mathcal{L}_{(d_1,\ldots,d_{n-1},k)}(m_1, \ldots, m_s)$ and $\mathcal{L}_2 = \mathcal{L}_{(d_1,\ldots,d_{n-1},d_n-k)}$ $(m_{s+1}, \ldots, m_r)$ with $k \leq d_n$ and $s \leq r$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are fiber non-special and hold certain conditions 2.2.1, then $\mathcal{L}$ is fiber non-special.*

In Chapter 3 we study normal algebraic varieties endowed with an effective action of an algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$, the so-called $\mathbb{T}$-varieties. Given a $\mathbb{T}$-variety X we define its *complexity* as $\dim X - \dim \mathbb{T}$. Such $\mathbb{T}$-varieties admit a combinatorial description starting with the well-known case of complexity-zero $\mathbb{T}$-varieties, i.e., toric varieties (see e.g. [**20**] and [**31**, Chapter 1]). Then, the complexity-one case was systematically studied in [**31**, Chapter 2 and 4], [**45**] and [**22**]. Finally, in [**3**, **4**] a combinatorial description is provided for arbitrary $\mathbb{T}$-varieties.

The topology of $\mathbb{Q}$-factorial complete toric varieties has been well studied, for instance in the books [**16**, Chapter 12] and [**23**, Chapter 5]. In particular, the fundamental group, the cohomology groups, the cohomology ring and the Chow ring of toric varieties are known in different degrees of generality. In Chapter 3 we generalize those results to the case of $\mathbb{T}$-varieties. In the toric setting, these objects depend on the combinatorial and geometric structure of its defining fan, while in higher complexity they also depend on the topology of a normal semi-projective variety Y of dimension $\dim X - \dim \mathbb{T}$, which is a kind of quotient (chow quotient) of X by the torus action.

In Section 1.3 we recall the language of divisorial fans to describe $\mathbb{T}$-varieties. In particular, a divisorial fan $\mathcal{S}$ living on (Y, N), where Y is an algebraic variety of dimension k and $N_{\mathbb{Q}}$ a n-dimensional $\mathbb{Q}$-vector space, describes a $\mathbb{T}$-variety $X(\mathcal{S})$ of dimension $n+k$ and complexity k. To $\mathcal{S}$ we associate a $\mathbb{T}$-variety $\widetilde{X}(\mathcal{S})$ which admits a categorical quotient $\pi\colon \widetilde{X}(\mathcal{S}) \to Y$ and a contraction morphism $\widetilde{X}(\mathcal{S}) \to X(\mathcal{S})$ (see Section 1.3). Moreover $\mathcal{S}$ defines a finitely generated abelian group $N(\mathcal{S})$. The fundamental group of $X(\mathcal{S})$ is given by the following.

THEOREM 3. *Let $X(\mathcal{S})$ be a complete $\mathbb{T}$-Variety with log-terminal singularities. Then we have an isomorphism*

$$\pi_1(X(\mathcal{S})) \simeq N(\mathcal{S}) \times \pi_1(Y).$$

Our next main result concerns the rational Chow ring $A^*(\widetilde{X}(\mathcal{S}))_{\mathbb{Q}}$ of the complexity-one $\mathbb{T}$-variety $\widetilde{X}(\mathcal{S})$ for a divisorial fan $\mathcal{S}$ on $Y = \mathbb{P}^1$. The theorem holds for a class of complete and simplicial divisorial fans called shellable, introduced in Definition 3.2.9. This notion generalizes the usual shellable condition for fans. Given a divisorial fan $\mathcal{S}$ on $(\mathbb{P}^1, N_{\mathbb{Q}})$, in Section 3.2 we construct the polynomial ring $\mathbb{Q}[D : D \text{ invariant divisor}]$ and the ideal I generated by all the linear relations between the invariant divisors, plus the monomials corresponding to subsets of invariants divisors with empty intersection (see Notation 3.2.10).

THEOREM 4. *Let $\mathcal{S}$ be a complete, simplicial and shellable divisorial fan on $\mathbb{P}^1$. Then we have an isomorphism*

$$\mathbb{Q}[D : D \text{ invariant divisor}]/I \to A^*(\widetilde{X}(\mathcal{S}))_{\mathbb{Q}}, \qquad D + I \mapsto [D].$$

Finally, we study the canonical map from rational Chow groups to rational Borel-Moore homology groups of complexity-one $\mathbb{T}$-varieties coming from divisorial fans satisfying the conditions of Definition 3.2.9 and prove the following.

THEOREM 5. *Let $\mathcal{S}$ be a complete, simplicial and shellable divisorial fan on $\mathbb{P}^1$. Then the canonical map from Chow groups to Borel-Moore homology*

$$A_k(\widetilde{X}(\mathcal{S}))_{\mathbb{Q}} \to H_{2k}(\widetilde{X}(\mathcal{S}); \mathbb{Q}),$$

is an isomorphism.

The thesis is organized as follows. In Chapter 1 we introduce the necessary background for Chapter 2 and 3. First in Section 1.1 we introduce algebraic varieties, divisors and sheaf cohomomology, which are the basic concepts appearing in both chapters. Then, in Section 1.2 we introduce linear systems in algebraic varieties which is the main concept in Chapter 2. In Section 1.3 we have two subsections, the former with the background of topology of Chapter 3 and the latter with the language of divisorial fans to describe $\mathbb{T}$-varieties. In Section 1.4 we introduce schemes and show that varieties are in fact schemes. Finally, in Section 1.5 we introduce toric varieties via fans and give an example of a complexity-one $\mathbb{T}$-variety. In Chapter 2 and 3, every section is devoted to the proof of the corresponding theorem.

CHAPTER 1

Preliminaries

In this chapter we discuss the background of Chapter 2 and 3. We start in Section 1.1 with a brief introduction to algebraic varieties and Weil divisors, which are the basic concepts appearing in both chapters. Then in section 1.2 and 1.3 we specialize to the specific background of Chapter 2 and 3, respectively. In particular, section 1.2 introduces linear systems in projective varieties and Cox rings, and Section 1.3 introduces $\mathbb{T}$-Varieties via divisorial fans and topology of algebraic varieties.

1.1. Algebraic Varieties and Divisors

In this section we introduce Algebraic Varieties, Regular and rational functions, Divisors and Sheaf cohomology. We use the notation of [**26**, Chapter 1]. We work over an algebraically closed field $\mathbb{K}$.

Definition 1.1.1 (Affine varieties). We denote by $\mathbb{A}^n$ the *affine n-space*. We say that a subset $Z \subset \mathbb{A}^n$, is an *algebraic subset* if Z can be expressed as the common zeros of a finite set of polynomials $f_1, \dots, f_r \in \mathbb{K}[x_1, \dots, x_n]$, or equivalently, as the zero set of the ideal $I = \langle f_1, \dots, f_r \rangle$. We define the *Zariski topology* on $\mathbb{A}^n$ by taking the closed subsets to be the algebraic subsets. We say that a subset $X \subset \mathbb{A}^n$ is a *affine algebraic variety* (or *affine variety*) if X is the zero set of a prime ideal $I \subset \mathbb{K}[x_1, \dots, x_n]$, with the induced Zariski topology. In $\mathbb{A}^n$ every closed subset is a finite union of affine algebraic varieties no one containing another. Given an affine variety X defined by an ideal I, we denote by $A(X) := \mathbb{K}[x_1, \dots, x_n]/I$, its *coordinate ring*. The field of fractions of $A(X)$, denoted by $\mathbb{K}(X)$, is calle the *field of rational functions of X*. We define the *dimension* of X as the supremum of all integers n such that there exists a chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X \subset \mathbb{A}^n$, of affine varieties. An open subset of an affine variety is called a *quasi-affine variety*. Given a quasi-affine variety X, and a function $f\colon X \to \mathbb{K}$, we say that f is *regular* at a point $p \in X$, if there is an open neighbourhood $p \in U \subset X$, and polynomials $g, h \in \mathbb{K}[x_1, \dots, x_n]$, such that h is nowhere zero on U, and $f = \frac{g}{h}$ on U. We say that f is regular on X if it is regular at every point.

Definition 1.1.2 (Projective varieties). We denote by $\mathbb{P}^n_{\mathbb{K}}$ (or simply $\mathbb{P}^n$) the *projective n-space over* $\mathbb{K}$ and we say that a subset $Z \subset \mathbb{P}^n$, is an *algebraic subset* if Z is the zero set of an homogeneous ideal of $\mathbb{K}[x_0, \dots, x_n]$. The *Zariski topology* of $\mathbb{P}^n$ is defined by taking the closed subsets to be the algebraic subsets. A *projective algebraic variety* (or *projective variety*) is the zero set of a homogeneous prime ideal of $\mathbb{K}[x_0, \dots, x_n]$. Given $X \subset \mathbb{P}^n$ a projective variety and I its defining ideal, we denote by $S(X) := \mathbb{K}[x_0, \dots, x_n]/I$ its *homogeneous coordinate ring*. The field of fractions of $S(X)$, denoted by $\mathbb{K}(X)$, is calle the *field of rational functions of X*. We define the *dimension* of X as the supremum of all integers n such that there exists a chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X \subset \mathbb{P}^n$, of projective varieties. An open subset of a projective variety is called a *quasi-projective variety.* Given a quasi-projective variety X, and a function $f\colon X \to \mathbb{K}$, we say that f is *regular* at a point $p \in X$, if there is an open neighbourhood $p \in U \subset X$, and homogeneous polynomials $g, h \in \mathbb{K}[x_0, \ldots, x_n]$ of the same degree, such that h is nowhere zero on U, and $f = \frac{g}{h}$ on U. We say that f is regular on X if it is regular at every point.

Definition 1.1.3 (Varieties). A *variety over* $\mathbb{K}$ (or a *variety*), is an affine, quasi-affine, projective, or quasi-projective variety. If X and Y are two varieties, a function $\phi\colon X \to Y$, is said to be a *morphism* if it is a continuous map such that for every open set $V \subset Y$, and regular function $f\colon V \to \mathbb{K}$, the function $f \circ \phi\colon \phi^{-1}(V) \to \mathbb{K}$ is regular. A morphism $\phi\colon X \to Y$ is an *isomorphism* if there exists and inverse morphism $\psi\colon Y \to X$ with $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_Y$. A *subvariety* of a variety X is a subset $Y \subset X$ which is a variety with the induced topology. The *codimension* of a subvariety $Y \subset X$ is the supremum of all integers n such that there exists a chain $Y \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$, of subvarieties of X. The product of two varieties X and Y is also a variety, we denote such variety by $X \times Y$.

Definition 1.1.4 (Topological spaces). In a topological space X, we say that a nonempty subset Y is *irreducible* if it is not the union of two proper subsets $Y = Y_1 \cup Y_2$, each one closed in Y. A topological space X is called *noetherian* if any sequence of closed subsets $Y_1 \supset Y_2 \supset \ldots$ eventually stops. A topological space X is called *quasi-compact* if for every open cover there exists a finite subset of open sets which cover X. Observe that varieties are irreducible and noetherian.

Definition 1.1.5 (Regular and rational functions). Let X be a variety. We denote by $\mathcal{O}(X)$ the ring of all regular functions on X.

If $p \in X$, we define the *local ring of* p *on* X, denoted by $\mathcal{O}_{p,X}$, to be the ring of germs of regular functions on X in a neighbourhood of p. In fact, the elements of the local ring of p on X are pair (U, f), where U is an open subset of X containing p, and f is a regular function on U. We identify two elements (U, f) and (V, g) if $f = g$ on $U \cap V$. In fact, $\mathcal{O}_{p,X}$ is a local ring, its unique maximal ideal is $m_{p,X}$, generated by pairs (U, f), with f vanishing at p.

The *field of rational functions of* X, denoted by $\mathbb{K}(X)$, is defined as follows: an element is a pair (U, f), where U is a nonempty set of X an f is a regular function on U, and we identify two elements (U, f) and (V, g) if $f = g$ on $U \cap V$.

Given two varieties X and Y, a *rational map* $f\colon X \to Y$, between two varieties is an equivalence of pairs (f_U, U) in which f_U is a morphism of varieties from an open subset $U \subset X$ to Y, and any two such pairs (f_U, U) and (f_V, V) are considered equivalent if f_U and f_V coincide in $U \cap V$. f is said to be *birational* if there exists a rational map $g\colon Y \to X$, which is its two-sided inverse.

Definition 1.1.6 (Discrete valuation ring). Given a field K and a G a totally ordered abelian group. A *valuation* of K with values in G is a map $\upsilon\colon K - \{0\} \to G$, with the following properties

$$\upsilon(xy) = \upsilon(x) + \upsilon(y), \qquad \forall x, y \in K, x, y \neq 0,$$

$$\upsilon(x + y) \geq \min(\upsilon(x), \upsilon(y)), \qquad \forall x, y \in K, x, y \neq 0.$$

If υ is a valuation, then the set $R := \{x \in K \mid \upsilon(x) \geq 0\} \cup \{0\}$, is a subring of K, which we call the *valuation ring of* υ. If G is the integers, the corresponding valuation ring is called a *discrete valuation ring*.

DEFINITION 1.1.7 (Regular local rings). We say that a ring R is *noetherian* if it satisfies the ascending chain condition for prime ideals. It is, for any sequence $I_1 \subsetneq I_2 \subsetneq \dots$ of prime ideals, there exists $r \in \mathbb{Z}_{\geq 0}$, such that $I_r = I_{r+1} = \dots$. In particular, the coordinate ring (resp. homogeneous coordinate ring) of an affine (resp. projective variety) is noetherian.

Let R be a ring and $I \subset R$ a prime ideal, we define the *height of* I to be the supremum of all integers n such that there exists a chain $I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = I$, of prime ideals. The *dimension of* R is the supremum of the height of all the ideals. Observe that if Y is an affine variety its dimension equals the dimension of its coordinate ring, and if Y is a projective variety its dimension equals the dimension of its coordinate rings minus one.

Observe that if Y is an affine or projective variety its dimension equals the dimension of its coordinate ring (or homogeneous coordinate ring).

Let R be a noetherian local ring, with maximal ideal m and residue field $k = R/m$. We say that R is a *regular local ring* if $\dim_k m/m^2 = \dim A$. A *local homomorphism* is a homomorphism $R \to S$, of local rings, such that the preimage of the maximal ideal of S is the maximal ideal of R.

DEFINITION 1.1.8 (Smooth varieties). Given a variety X. We say that X is *nonsingular* at a point $p \in X$, if the local ring $\mathcal{O}_{p,X}$ is a regular local ring. X is *nonsingular* (or *smooth*) if it is nonsingular at any point, otherwise it is called *singular*. A variety X is *normal* at a point $p \in X$, if the local ring $\mathcal{O}_{p,X}$ is integrally closed in its field of fractions. We say that X is *normal* if it is normal at any point. Observe that any smooth variety is normal. Given a variety X, the subset of points of X where the local ring is not regular is called *singular locus of* X. If X is a normal variety, then its singular locus has codimension at least two.

DEFINITION 1.1.9 (Divisors). Let X be a noetherian integral separated scheme which is regular in codimension one (see 1.4). We say that a point $p \in X$ is a *generic point* if its closure is X. Given a subscheme $Y \subset X$, we say that a point $p \in X$ is a *generic point of* Y if its closure is Y. A *prime divisor* on X is a closed integral subscheme Y of codimension one. A *Weil divisor* (or simply a *divisor*) of X is a finite formal sum $\sum n_i Y_i$, where Y_i are prime divisors and n_i integers. We denote by $\mathrm{WDiv}(X)$, the free abelian group generated by the Weil divisors. A divisor is said to be *effective* if $n_i \geq 0$, for each i. We write $D \geq 0$ if a divisor is effective. The *support* of a Weil divisor $D = \sum n_i Y_i$ is the closed subset $\mathrm{supp}(D) := \cup_i Y_i \subset X$. Given a prime divisor $Y \subset X$, and $\eta \in Y$ a generic point, the local ring $\mathcal{O}_{\eta,X}$ is a discrete valuation ring. We call the corresponding discrete valuation υ_Y the *valuation* of Y.

Given $f \in \mathbb{K}(X)$, we define the divisor of f, denoted (f), by

$$(f) = \sum \upsilon_Y(f) Y,$$

where the sum is taken over all prime divisors of X. Any divisor of this form is called a *principal divisor*. The subgroup of $\mathrm{WDiv}(X)$ freely generated by principal divisors is denoted by $\mathrm{PDiv}(X)$. We say that two Weil Divisors D and D' are linearly equivalent, denoted by $D \sim D'$, if $D - D'$ is a principal divisor. We denote

$\mathrm{Cl}(X) := \mathrm{WDiv}(X)/\,\mathrm{PDiv}(X)$, the *Divisor class group* of X. Recall by 1.4.15 that this definition also works for algebraic varieties.

Definition 1.1.10 (Sheaves). Given a topological space X, we define a *presheaf* $\mathcal{F}$ of abelian groups on X is given by the following data

- for every open subset $U \subset X$, an abelian group $\mathcal{F}(U)$,
- for every inclusion $V \subset U$, of open subsets of X, a morphism of abelian group $\rho_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$,

with the following conditions,

- $\mathcal{F}(\emptyset) = 0$,
- $\rho_{U,U}$ is the identity map on $\mathcal{F}(U)$, for every open subset U of X, and
- if $W \subset V \subset U$ is an inclusion of open subsets of X, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

Given a morphism $\rho_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$, and an element $s \in \mathcal{F}(U)$, we denote by $s|_V$ the element $\rho_{U,V}(s)$. The group of sections $\mathcal{F}(X)$ of a presheaf $\mathcal{F}$ on X is also denoted by $\Gamma(X, \mathcal{F})$. A presheaf $\mathcal{F}$ on X is calle a *sheaf* if it satisfies the following conditions

- if U is an open set of X, $\{V_i\}_{i \in I}$ an open covering of U and $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all $i \in I$, then $s = 0$.
- if U is an open set of X, $\{V_i\}_{i \in I}$ an open covering of U and $s_i \in \mathcal{F}(V_i)$ for each $i \in I$, with the property that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for each i, j. Then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i.

The *presheaves* and *sheaves* of rings and modules are defined analogously. The *support of a sheaf* $\mathcal{F}$ is the set of points $p \in X$ such that $\mathcal{F}_p \neq 0$. Given a sheaf $\mathcal{F}$ on a topological space X, and a point $p \in X$, we denote the *stalk of* $\mathcal{F}$ *at* p to be $\mathcal{F}_p = \lim_{x \in U} \mathcal{F}(U)$, the direct limit indexed over all open subsets of X containing p.

Definition 1.1.11 (Morphism of sheaves). Given two presheaves $\mathcal{F}$ and $\mathcal{G}$ of abelian groups on X, a *morphism of sheaves* $\phi \colon \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelian groups $\phi(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$, for each open subset $U \subset X$, such that for every inclusion of open subsets $V \subset U$, the following diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{U,V}} & & \downarrow{\scriptstyle \rho'_{U,V}} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V), \end{array}$$

is commutative, where ρ and ρ' are the restriction maps of $\mathcal{F}$ and $\mathcal{G}$ respectively. An *isomorphism* is a morphism which has a two-sided inverse. Given a morphism of sheaves of X, $\mathcal{F} \to \mathcal{G}$ and a point $p \in X$, then there is an induced morphism of the stalks $\mathcal{F}_p \to \mathcal{G}_p$, obtained by direct limit.

Definition 1.1.12 (Sheafification). Let $\mathcal{F}$ be a presheaf on a topological space X. The *sheafification of* $\mathcal{F}$ or the *sheaf associated to the presheaf* $\mathcal{F}$ is a sheaf $\mathcal{F}'$ defined by the data:

$$\begin{aligned} \mathcal{F}'(U) \quad = \quad & \{\phi = (\phi_p)_{p \in U} \mid \phi_p \in \mathcal{F}_p, \text{ for all } p \in U, \\ & \text{such that for all } q \in U, \text{ there is a neighborhood } q \in V \subset U \\ & \text{and a section } \phi' \in \mathcal{F}(V), \text{ with } \phi'_p = \phi_p \text{ for all } p \in V\}. \end{aligned}$$

It is standard to check that $\mathcal{F}'$ is in fact a sheaf. If $\mathcal{F}$ is a sheaf, then $\mathcal{F} = \mathcal{F}'$. The stalks of $\mathcal{F}$ and $\mathcal{F}'$ agrees in each point.

DEFINITION 1.1.13 (Kernel of morphisms of sheaves). Let $\phi\colon \mathcal{G} \to \mathcal{F}$ be a morphism of presheaves. We define the *presheaf kernel* and *presheaf image* of ϕ to be $U \mapsto \ker(\phi(U))$ and $U \mapsto \operatorname{im}(\phi(U))$ respectively. A *subsheaf* of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{G}$, such that for each open set U, $\mathcal{G}(U)$ is a subgroup of $\mathcal{F}(U)$. The *quotient sheaf* of a sheaf $\mathcal{F}$ by a subsheaf $\mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$.

EXAMPLE 1.1.14. Given a variety X. For every open subset $U \subset X$, let $\mathcal{O}_X(U)$ be the ring of regular functions of U, and given an inclusion of open subsets $V \subset U$, let $\rho_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ be the restriction map. Then $\mathcal{O}_X$ is a sheaf on X, called the *structure sheaf of* X.

We say that a sheaf $\mathcal{F}$ on a variety X is a sheaf of $\mathcal{O}_X$-modules, if for each open subset $U \subset X$, $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module. We say that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is *locally free* if for each point $p \in X$, there exists an open neighbourhood U of p such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X(U)$-module. Observe that in this case, every stalk $\mathcal{F}_p$, with $p \in X$ is a free $(\mathcal{O}_X)_p$-module. Moreover, the stalk $(\mathcal{O}_X)_p$ is canonically isomorphic to the local ring $\mathcal{O}_{X,p}$, then $\mathcal{F}_p$ is a $\mathcal{O}_{X,p}$-module. If $\mathcal{F}_p$ is of finite rank n for every $p \in X$, we say that $\mathcal{F}$ is a *locally free $\mathcal{O}_X$-module of rank n*. Given two sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a variety X, we define a sheaf of $\mathcal{O}_X$-modules denoted by $\mathcal{F} \otimes \mathcal{G}$ and given by

$$(\mathcal{F} \otimes \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

with the induced restriction maps.

EXAMPLE 1.1.15. Given a Weil divisor D of a variety X we associate a sheaf of $\mathcal{K}$-vectorial spaces on X, denoted by $\mathcal{O}_X(D)$ and given by

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{K}(X) \mid ((f) + D)|_U \geq 0\},$$

where the restriction $D|_U$ is defined as 0 if $D \cap U = \emptyset$ and $D \cap U$ otherwise. Given two divisors D_1 and D_2 of X, we have that $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$ if and only if $D_1 \sim D_2$. We say that a Weil divisor D of X is *Cartier* if there exists an open cover $\{U_i\}_{i\in I}$ such that $D|_{U_i}$ is a principal divisor for each i. Moreover, we say that a divisor D is $\mathbb{Q}$-*Cartier* if some multiple is Cartier. A Cartier divisor D on a variety X can be represented by an open cover $\{U_i\}_{i\in I}$ by affine subsets of X and rational functions f_i defined in U_i, such that for each $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, the function $f_i \circ f_j^{-1}$ is regular and has no zeros in $U_i \cap U_j$. The subgroup of $\operatorname{WDiv}(X)$ freely generated by effective Cartier divisors is denoted by $\operatorname{CaDiv}(X)$. The group of Cartier divisors modulo linear equivalence is denoted by $\operatorname{CaCl}(X)$. Given a morphism of varieties $f\colon X \to Y$, and a Cartier divisor $D \subset Y$, defined by the rational functions $f_i \in U_i$, with $\{U_i\}_{i\in I}$ an open affine cover of Y, we define the pull-back $f^*(D)$ to be the Cartier divisor of X defined by the rational functions $f_i \circ f \in f^{-1}(U_i)$. This extends to a homomorphism $f^*\colon \operatorname{CaDiv}(Y) \to \operatorname{CaDiv}(X)$.

DEFINITION 1.1.16 (Factorial variety). A variety X is said to be *factorial* if every divisor is Cartier. Also, a variety X is said to be $\mathbb{Q}$-*factorial* if every divisor is $\mathbb{Q}$-Cartier. Observe that a variety X is factorial if all its local rings are unique factorization domains.

Example 1.1.17. Given X a variety, and G an abelian group we can define the *constant presheaf with value* G, denoted by $\mathcal{G}$, to be the sheaf given by the data: $\mathcal{G}(U) = G$, for any open subset $U \subset X$, and $\rho_{U,V} : \mathcal{G}(U) \to \mathcal{G}(V)$, is the identity on G for each $V \subset U$. This presheaf induces a sheaf $\mathcal{G}$ whose stalks are all equal to G.

Definition 1.1.18 (Coherent sheaf). A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a variety X is said to be *coherent* if the following conditions holds:

- $\mathcal{F}$ is of *finite type*. It is, for any points $p \in X$ there exists a neighbourhood $U \subset X$ of x, and a surjective morphism $\mathcal{O}_X(U)^n \to \mathcal{F}(U)$, of $\mathcal{O}_X$-modules, for some n.
- For any open subset $U \subset X$, any $n \in \mathbb{Z}_{\geq 0}$ and any morphism $\phi\colon \mathcal{O}_X(U)^n \to \mathcal{F}(U)$, of $\mathcal{O}_X$-modules, the kernel of ϕ is of finite type.

Definition 1.1.19 (Contraction morphism). Let X and Y be algebraic varieties. A morphism $f\colon X \to Y$ is said to be a *contraction* if there exists a proper closed subvariety $V \subset Y$ such that $\dim(f^{-1}(V)) > \dim(V)$ and $f|_{X\setminus f^{-1}(V)}\colon X \setminus f^{-1}(V) \to Y \setminus V$ is an isomorphism.

Definition 1.1.20 (Picard group). Given a variety X, a locally free $\mathcal{O}_X$-module of rank 1 is called a *invertible sheaf.* If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves, then so is $\mathcal{L} \otimes \mathcal{M}$. Moreover, given a invertible sheaf $\mathcal{L}$, there exists a invertible sheaf $\mathcal{L}^{-1}$ on X such that $\mathcal{L} \otimes \mathcal{L}^{-1} \simeq \mathcal{O}_X$.

The *Picard group* of X is the group of isomorphism classes of invertible sheaves on X. On any variety X, the map $D \mapsto \mathcal{O}_X(D)$ gives an isomorphism of the group $\mathrm{CaCl}(X)$ of Cartier divisors modulo linear equivalence to $\mathrm{Pic}(X)$ (see [**26**, Proposition 6.15]).

Definition 1.1.21 (Sheaf Cohomology). Let X be a variety, $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules and $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of X by affine varieties. Fix a well-ordering of the index set I. For any finite set of indices $i_0, \dots, i_p \in I$ we denote the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0,\dots,i_p}$. We define a complex $\mathcal{C}^{\cdot}(\mathcal{U}, \mathcal{F})$ of abelian groups as follow. For each $p \geq 0$, let

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0,\dots,i_p}).$$

We denote an element α of $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ by elements $\alpha_{i_0,\dots,i_p} \in \mathcal{F}(U_{i_0,\dots,i_p})$, for each $(p+1)$-tuple $i_0 < \cdots < i_p$ of elements of I. We define a co-boundary map $d\colon \mathcal{C}^p \to \mathcal{C}^{p+1}$, by setting

$$(d\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k a_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}|_{U_{i_0}},\dots,i_{p+1}}.$$

Where the notation $\widehat{i_k}$ means that we omit the term i_k. So, we have a complex of abelian groups. We define the cohomology of $\mathcal{F}$ as

$$H^p(X, \mathcal{F}) := h^p(\mathcal{C}^{\cdot}(\mathcal{U}, \mathcal{F})).$$

1.2. Linear Systems on Algebraic Varieties

In this section we introduce linear systems, which is the central concept of Chapter 2. We work over an algebraically closed field $\mathbb{K}$ and consider projective smooth varieties.

DEFINITION 1.2.1 (Linear Systems). Let X be a projective smooth variety and D_0 a Weil divisor. The *complete linear system of* D_0 is $|D_0| := \{D \mid D \geq 0, D \sim D_0\}$. Observe that for each nonzero section $s \in \Gamma(X, \mathcal{O}_X(D_0))$, the divisor $\mathrm{div}(s)+D_0$ is an element of $|D_0|$ and every element of $|D_0|$ occurs in this way, we call this element the *divisor of zeros of* s. Moreover, two sections $s, s' \in \Gamma(X, \mathcal{O}_X(D_0))$ have the same divisor of zeros if and only if there exists $\lambda \in \mathbb{K}$ such that $s' = \lambda s$. Then, there is a one-to-one correspondence

$$(\Gamma(X, \mathcal{O}_X(D_0)) - \{0\})/\mathbb{K}^* \to |D_0|, \qquad s \mapsto \mathrm{div}(s) + D_0.$$

This gives $|D_0|$ the structure of a projective space over $\mathbb{K}$.

A linear system $\mathcal{L}$ on X is a subset of a complete linear system $|D_0|$ which is a linear subspace considering the projective space structure of $|D_0|$. We say that $\mathcal{L}$ is a *sub-linear* system of $|D_0|$. The *dimension* of $\mathcal{L}$ is its dimension as a projective space. The *Base locus* of a linear system $\mathcal{L}$ is the intersection of all the supports of its divisors. We denote the Base locus of $\mathcal{L}$, by $\mathrm{Bs}(\mathcal{L})$. Given a divisor D, we define its *Base locus* $\mathrm{Bs}(D)$, to be the base locus of $|D|$.

Given a linear system $\mathcal{L}$ on X, we can give $\mathrm{Bs}(\mathcal{L})$ scheme structure (see 1.4.2) as follows: Assume $\mathcal{L}$ corresponds to the linear subspace $V \subset H^0(X, D)$, where D is a divisor on X. We have a morphism of sheaves $V \otimes_{\mathbb{K}} \mathcal{O}_X \to \mathcal{L}$, determined by evaluation in V, which induces a morphism $V \otimes_{\mathbb{K}} \mathcal{L}^* \to \mathcal{O}_X$, whose image is a sheaf ideal determining a subscheme of X with topological space $\mathrm{Bs}(\mathcal{L})$.

DEFINITION 1.2.2 (Blowing-up at points). Let X be a k-dimensional smooth projective variety and $p \in X$ a point. The blow-up of X at p is a smooth projective variety $\widetilde{X}$ with a surjective morphism $\pi\colon \widetilde{X} \to X$, such that $E := \pi^{-1}(p) \simeq \mathbb{P}^{k-1}$ and $\pi|_{\widetilde{X}-E}\widetilde{X} - E \to X - \{p\}$ is an isomorphism. The blow-up of X at a point always exists, and is unique up to isomorphism. We say that $\widetilde{X}$ is obtained from X by *blowing-up* p. E is a prime divisor of $\widetilde{X}$ and is called the *exceptional divisor of* $\widetilde{X}$. The divisor class group of $\widetilde{X}$ is isomorphic to $\mathrm{Cl}(X) \oplus \langle E \rangle$, where $\langle E \rangle$ denotes the subgroup generated by the class of E in $\mathrm{Cl}(\widetilde{X})$. Given r different points $p_1, \ldots, p_r \in X$, we define the *blow-up of* X *at the points* $p_1, \ldots, p_r$ to be the smooth projective variety $\mathrm{Bl}_{p_1,\ldots,p_r} X$ obtained by blowing-up those points inductively. In fact, we obtain a surjective morphism $\pi\colon \mathrm{Bl}_{p_1,\ldots,p_r} X \to X$, which fiber over a point p_i is isomorphic to $\mathbb{P}^{k-1} =: E_i$ and called the *i-th exceptional divisor* and the restriction of π to the complement of $\cup_{i=1}^r E_i$ is an isomorphism. By induction, it follows that

$$\mathrm{Cl}(\mathrm{Bl}_{p_1,\ldots,p_r} X) \simeq \mathrm{Cl}(X) \oplus_{i=1}^r \langle E_i \rangle.$$

DEFINITION 1.2.3 (Linear systems of hypersurfaces passing throught fixed points). Let X be a smooth projective variety as above. Fix r different points $p_1, \ldots, p_r \in X$ and a class $[D] \in \mathrm{Cl}(X)$. Given r nonnegative integers $m_1, \ldots, m_r$ we define the linear system of hypersurfaces of degree $[D]$ passing through p_i with multiplicity m_i for each i, to be the complete linear system $|\pi^*D - \sum_{i=1}^r m_i E_i|$ on $\mathrm{Bl}_{p_1,\ldots,p_r} X$. This linear system is also a sub-linear system of $|D|$ on X. Observe that the dimension of such linear system depends on the points $p_1, \ldots, p_r$. We say that $p_1, \ldots, p_r$ are in *very general position* if for every $[D] \in \mathrm{Cl}(X)$ and $m_1, \ldots, m_r$ nonnegative integers, we have that $\dim(\mathcal{L}_{[D]}(m_1, \ldots, m_r))$ attains its minimum.

DEFINITION 1.2.4 (Cox sheaf and Cox ring). Let X be a smooth projective variety with freely finitely generated Divisor class group. Fix a subgroup $K \subset$

WDiv(X), such that the canonical map $K \to \mathrm{Cl}(X)$, sending $D \in K$ to $[D]$ is an isomorphism. We define the *Cox sheaf associated with* K to be the following sheaf of rings

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}_X(D),$$

where $D \in K$ represents $[D] \in \mathrm{Cl}(X)$ and the multiplication in $\mathcal{R}$ is defined by multiplying homogenous sections in $\mathbb{K}(X)$. Up to isomorphism, the sheaf $\mathcal{R}$ does not depend on K (see [**2**, 1.4.1.1]). The *Cox ring of* X is the ring of global sections of $\mathcal{R}$. We denote the Cox ring of X by

$$\mathrm{Cox}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D)).$$

For a complete construction of Cox rings, basic properties and selected topics see [**2**].

Observe that given a projective smooth variety X, a class $[D] \in \mathrm{Cl}(X)$, points $p_1, \ldots, p_r$ and nonnegative integers $m_1, \ldots, m_r$, the problem of calculating the dimension $\dim(\mathcal{L}_{[D]}(m_1, \ldots m_r))$ is equivalent to calculate the dimension, as a $\mathbb{K}$-vectorial space, of the homogeneous part of degree $[D]$ of $\mathrm{Cox}(\mathrm{Bl}_{p_1,\ldots,p_r} X)$.

1.3. Topology and Varieties with Torus Action

Subsection 3.1: Topological objects

We start introducing some concepts of topology applied to affine and projective varieties. In this section we work over the complex numbers $\mathbb{C}$. We define the fundamental group, cohomology ring and Chow ring of an algebraic variety, which are algebraic objects containing the topological information of the variety (with its analytical topology). In Chapter 3 we compute those algebraic objects for some classes of $\mathbb{T}$-varieties.

Definition 1.3.1 (Homotopy). Let f and g two continuous functions from a topological space X to a topological space Y. A *homotopy* between f and g is a continuous function $H\colon X \times [0,1] \to Y$, such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. If such function exists, we say that f and g are *homotopic*. Homotopy is an equivalence relation. A standard reference for homotopy is [**27**, Chapter 0].

Recall that any variety over the complex numbers has an *Analytical topology* induced by the euclidean topology of $\mathbb{C}^n$ and $\mathbb{P}^n_{\mathbb{C}}$. Let X be a variety with its analytical topology, and let x_0 be a point of X. We say that a continuous function $f\colon [0,1] \to X$ is a *loop* with *base points* x_0 if $f(0) = x_0 = f(1)$. The *fundamental group* $\pi_1(X, x_0)$ *of* X *with base point* x_0 is the set of loops modulo homotopy, endowed with the multiplication define by

$$(f \cdot g)(t) := \begin{cases} f(2t), & 0 \le t \le \frac{1}{2}, \\ g(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Since varieties are path-connected, for any two different points x_0 and x_1 in X, the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic, so we simply denote this group by $\pi_1(X)$. Given a morphism of algebraic varieties $\phi\colon X \to Y$, it induces a homomorphism between its fundamental groups $\phi^*\colon \pi_1(X) \to \pi_1(Y)$. A standard reference for fundamental group is [**27**, Chapter 1].

Definition 1.3.2 (Cohomology ring). Given a variety X and a ring R, we define the *k-th singular cohomology with coefficients in* R, denoted by $H^k(X,R)$, to be the k-th sheaf cohomology group with the constant sheaf of R.

Given $H^k(X,R)$ the cohomology groups of X with coefficients in a commutative ring, one can define the cup product, which is a map

$$H^k(X,R) \times H^l(X,R) \to H^{k+l}(X,R).$$

The cup product gives a multiplication structure on the direct sum of cohomology groups

$$H^\bullet(X,R) \simeq \bigoplus_{k\in\mathbb{Z}_{\geq 0}} H^k(X,R).$$

The ring $H^\cdot(X,R)$ is called the *Cohomology ring of* X *with coefficients in* R. A standard reference for Cohomology ring is [27, Chapter 3]

Definition 1.3.3 (Chow ring). Let X be a variety. a k-cycle on X is a finite formal sum $\sum n_i[V_i]$, where V_i are k-dimensional subvarieties of X, and the n_i integers. We denote by Z_kX the abelian group freely generated by k-cycles. For any subvariety W of dimension $k+1$ of X, and any rational function $\mathbb{C}(W)^*$, we define a k-cycle $[\operatorname{div}(r)]$ on X given by

$$[\operatorname{div}(r)] = \sum \operatorname{ord}_V(r)[V],$$

where the sum is taken over all the subvarieties of W of codimension one, and ord_V is the valuation associated to the local ring $\mathcal{O}_{V,W} = \mathcal{O}_{\eta,W}$, and $\eta \in V$ a generic point. A k-cycle α is said to be *rationally equivalent to* 0, denoted by $\alpha \sim 0$, if there is a finite number of $(k+1)$-dimensional subvarieties W_i of X and $r_i \in \mathbb{K}(W_i)^*$, such that $\alpha = \sum[\operatorname{div}(r_i)]$. We say that two cycles are *rationally equivalent* if its difference is rationally equivalent to 0. The group of k-cycles modulo rational equivalence on X is A_kX. The *Chow ring of* X is the direct sum

$$A^*(X) := \bigoplus_{k=0}^{\dim X} A^k(X),$$

where multiplication is given by

$$[Y]\cdot[Z] = [Y \cap Z].$$

This construction is functorial, given a morphism of varieties $f\colon X \to Y$, there is an associated homomorphism of Chow rings $f_*\colon A^*(Y) \to A^*(X)$. Given a variety X we define its *Chow ring with rational coefficients* to be $A^*(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. For a complete construction and properties of the Chow ring of a variety see [29, Chapter 1].

For every variety X, there is a canonical homomorphism $A^*(X) \to H^{2*}(X)$, which sends the class of a k-cycle $[V_i]$ to the class of V_i in $H^{2k}(X)$.

Definition 1.3.4 (Borel-Moore homology). Let X be a variety embedded into a smooth manifold M of dimension m, such that X is a retract of an open neighbourhood of itself. Then we define the Borel-Moore homology of X to be

$$H_i^{BM}(X) = H^{m-i}(M, M\setminus X),$$

where $H^{m-i}(M, M\setminus X)$, is the *relative cohomology*. Given a morphism of varieties $f\colon X \to Y$, there is an induced homomorphism of Borel-Moore homology groups

$f_*\colon H_i^{BM}(Y) \to H_i^{BM}(X)$. If X is compact, then the Borel-Moore homology coincides with the usual singular homology. Given $Y \subset X$ a closed subvariety and $U := X \setminus Y$ its complement, there is a long exact sequence

$$\cdots \to H_i^{BM}(Y) \to H_i^{BM}(X) \to H_i^{BM}(U) \to H_{i-1}^{BM}(F) \to \ldots$$

This exact sequence comes from excision long exact sequence of relative cohomology. For more equivalent definitions and properties of Borel-Moore homology see [11, Chapter 5].

Subection 3.2: $\mathbb{T}$-Varieties via divisorial fans

In the well-known theory of toric varieties, a toric variety corresponds to a fan in a finite dimensional $\mathbb{Q}$-vector space. In this subsection we recall a generalization of this correspondence due to Altmann, Hausen and Süß [3, 4], see also [5] for a survey on known results about $\mathbb{T}$-varieties. Roughly speaking, a $\mathbb{T}$-variety of complexity k and dimension n (see 1.3.6) corresponds to a variety Y of dimension $n-k$ and a polyhedral divisor (see 1.3.8) living in a k-dimensional $\mathbb{Q}$-vector space and the variety Y. We introduce this description to generalize results of Fulton (see [23, Chapter 5]) about the topology of algebraic toric varieties to varieties of higher complexity in chapter 3. In appendix two, we recall the well-known correspondence between toric varieties and fans.

Definition 1.3.5 ($\mathbb{Q}$-divisors). Given a variety X, we define the group of $\mathbb{Q}$-Weil divisors to be $\operatorname{WDiv}_{\mathbb{Q}}(X) = \operatorname{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. A $\mathbb{Q}$-divisor D is said to be *Cartier* (resp. *effective*) if there exists an integer n, such that nD is a Cartier (resp. effective) Weil divisor.

Definition 1.3.6 ($\mathbb{T}$-Variety). We say that the action of a group G on a set X is *effective* if the unique element of G acting trivially is the identity. We denote by $\mathbb{T} = (\mathbb{C}^*)^n$ the *n-dimensional complex torus*. A *$\mathbb{T}$-variety* X is an algebraic variety endowed with an effective action of an algebraic torus $\mathbb{T}$. We define the *complexity* of the $\mathbb{T}$-variety as $\dim X - \dim \mathbb{T}$. If the complexity is zero, we say that X is a *toric variety*.

We fix N a finitely generated free abelian group and let $M := \operatorname{Hom}(N, \mathbb{Z})$ be the dual of N. We denote by $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ the associated rational vector spaces.

Definition 1.3.7 (σ-polyhedra). Let σ be a pointed polyhedral cone in $N_{\mathbb{Q}}$. For every convex polyhedron $\Delta \subseteq N_{\mathbb{Q}}$, we define its *tail cone* as $\operatorname{tail}(\Delta) = \{v \in N_{\mathbb{Q}} \mid v + \Delta \subseteq \Delta\}$ and we say that Δ is a σ-polyhedron. We denote by $\operatorname{Pol}^+_{\mathbb{Q}}(N, \sigma)$ the set of all σ-polyhedra in $N_{\mathbb{Q}}$, observe that this set endowed with the Minkowski sum is a semigroup. We will consider the empty set as a σ-polyhedron for any σ, with the addition rule $\emptyset + \Delta := \emptyset$.

Definition 1.3.8 (Polyhedral divisor). Let X be a normal projective variety. A *polyhedral divisor* on $(X, N_{\mathbb{Q}})$ is a formal sum $\mathcal{D} := \sum_D \Delta_D \otimes D \in \operatorname{Pol}^+_{\mathbb{Q}}(N, \sigma) \otimes_{\mathbb{Z}} \operatorname{CaDiv}(X)$, where Δ_D are convex σ-polyhedra in $N_{\mathbb{Q}}$ and at most finitely many coefficients Δ_D are different from σ. We call σ the tailcone of $\mathcal{D}$.

Definition 1.3.9 (Locus and restriction). Given a polyhedral divisor $\mathcal{D}$ on $(X, N_{\mathbb{Q}})$, we define its *open loci* to be $\operatorname{Loc}(\mathcal{D}) = X \setminus \cup_{\Delta_D = \emptyset} D$, its *support* to be $\operatorname{Loc}(\mathcal{D}) \cap \cup_{\Delta_D \neq \sigma} D$ and its *trivial locus* to be $U_{\mathcal{D}} = \operatorname{Loc}(\mathcal{D}) \setminus \operatorname{supp}(\mathcal{D})$.

Given an open subset $U \subseteq X$, we define the *restriction of the polyhedral divisor* $\mathcal{D}$ *to* U to be $\sum_{D \cap U \neq \emptyset} \Delta_D \otimes D|_U$ if $U \not\subseteq U_{\mathcal{D}}$ and $\sigma(\mathcal{D}) \otimes D$, where $D \in \mathrm{CaDiv}(U)$, otherwise.

Definition 1.3.10 (Intersection set). An *intersection set* of $\mathcal{D}$ is a non-empty intersection of divisors of $\mathrm{supp}(\mathcal{D})$. The intersection set is called *maximal intersection set* if it does not properly contain an intersection set. Given a maximal intersection set M the polyhedron $\sum_{M \subseteq D} \Delta_D$ is called a *maximal polyhedron* of $\mathcal{D}$. Given $y \in Y$, we denote by $\mathcal{D}_y$ the polyhedron $\sum_{y \in D} \Delta_D$.

Definition 1.3.11 (Evaluation of a polyhedral divisor). Let $\mathrm{CaDiv}_{\mathbb{Q}}(Y)$ be the group $\mathrm{CaDiv}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the set of rational $\mathbb{Q}$-Cartier divisors. Given a polyhedral divisor $\mathcal{D}$ on $(X, N_{\mathbb{Q}})$ with tail cone σ, we have an evaluation map into $\mathcal{D}\colon \sigma^\vee \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$, which we call *the evaluation map of* $\mathcal{D}$, and maps u in $\sum \min_{v \in \Delta_D} \langle u, v \rangle D$.

Definition 1.3.12 (p-divisor). Let $\mathcal{D}$ be a polyhedral divisor on $(Y, N_{\mathbb{Q}})$ with tail cone σ. Then $\mathcal{D}$ is a *p-divisor* if $\mathcal{D}(u)$ is a semiample divisor for every $u \in \sigma^\vee$ and $\mathcal{D}(u)$ is big for $u \in \mathrm{relint}(\sigma^\vee)$.

Definition 1.3.13 (Good quotient). Let G be an algebraic group acting on a variety X. A morphism $f\colon X \to Y$ is said to be *G-invariant* if it is constant along the orbits of the action. Moreover, f is called *affine* if the preimage of an open affine subset is affine. A morphism $f\colon X \to Y$ is called a *good quotient* for an action of G on X if f is affine, G-invariant and the $f^*\colon \mathcal{O}_Y \to (p_* O_X)^G$ is an isomorphism (here f^* is the pull-back morphism defined in 1.4.1, and $\mathcal{F}^G$ means the sheaf of algebras of G-invariants of $\mathcal{F}$).

Definition 1.3.14 (Affine $\mathbb{T}$-variety defined by a p-divisor). Given a p-divisor $\mathcal{D}$ on $(X, N_{\mathbb{Q}})$ with tail cone σ, we denote by $\widetilde{X}(\mathcal{D})$ the relative spectrum of the coherent sheaf of algebras $\mathcal{A}(\mathcal{D}) = \bigoplus_{u \in M \cap \sigma^\vee} \mathcal{O}_Y(\mathcal{D}(u))$ on $\mathrm{Loc}(\mathcal{D})$. There is a good quotient $\pi\colon \widetilde{X}(\mathcal{D}) \to \mathrm{Loc}(\mathcal{D})$ induced by the inclusion of sheaves $\mathcal{O}_Y \to \mathcal{A}(\mathcal{D})$. We define the *affine* $\mathbb{T}$*-variety defined by* $\mathcal{D}$ to be $X(\mathcal{D}) := \mathrm{Spec}\, H^0(\mathrm{Loc}(\mathcal{D}), \mathcal{A}(\mathcal{D}))$. This variety comes with a proper birational contraction (see 1.4.13 and 1.1.19) $\widetilde{X}(\mathcal{D}) \to X(\mathcal{D})$. The main result in [**3**] states that all normal affine $\mathbb{T}$-varieties arise this way.

Definition 1.3.15 (Intersection of p-divisors). Given two p-divisors $\mathcal{D}, \mathcal{D}'$ on $(Y, N_{\mathbb{Q}})$, we write $\mathcal{D}' \subseteq \mathcal{D}$ if $\Delta_{D'} \subseteq \Delta_D$ for each $D \in \mathrm{CaDiv}_{\geq 0}(Y)$. Observe that if $\mathcal{D}' \subseteq \mathcal{D}$, then we have an induced map $X(\mathcal{D}') \to X(\mathcal{D})$. We say that $\mathcal{D}'$ is a *face* if such map is an open embedding. The *intersection* $\mathcal{D} \cap \mathcal{D}'$ of $\mathcal{D}'$ and $\mathcal{D}$ is the polyhedral divisor $\sum_D (\Delta_D \cap \Delta'_D) \otimes D$.

Definition 1.3.16 (Divisorial fan). A *divisorial fan* $\mathcal{S}$ is a finite set of p-divisors on $(Y, N_{\mathbb{Q}})$ such that the intersection of any two p-divisors of $\mathcal{S}$ is a face of both and $\mathcal{S}$ is closed under taking intersections.

Definition 1.3.17 ($\mathbb{T}$-variety defined by a divisorial fan). Given a divisorial fan $\mathcal{S}$ on $(X, N_{\mathbb{Q}})$, we denote by $X(\mathcal{S})$ the $\mathbb{T}$-Variety obtained by gluing the affine $\mathbb{T}$-Varieties $X(\mathcal{D}^i)$ along the open subvarieties $X(\mathcal{D}^i \cap \mathcal{D}^j)$ for each $\mathcal{D}^i, \mathcal{D}^j \in \mathcal{S}$ and call it the $\mathbb{T}$*-variety defined by the divisorial fan* $\mathcal{S}$. The main result in [**4**] states that all normal $\mathbb{T}$-varieties arise this way.

The set $\{\sigma(\mathcal{D}) \mid \mathcal{D} \in \mathcal{S}\}$ form the so-called *tailfan* $\Sigma(\mathcal{S})$ of $\mathcal{S}$. The *locus* of $\mathcal{S}$ is the open set $\operatorname{Loc}(\mathcal{S}) := \cup_{\mathcal{D}\in\mathcal{S}} \operatorname{Loc}(\mathcal{D})$, the *support* of $\mathcal{S}$ is $\operatorname{supp}(\mathcal{S}) := \bigcup_{\mathcal{D}\in\mathcal{S}} \operatorname{supp}(\mathcal{D})$ and the *trivial locus* of $\mathcal{S}$ is the open set $U_{\mathcal{S}} := \bigcap_{\mathcal{D}\in\mathcal{S}} U_{\mathcal{D}}$. For any divisorial fan $\mathcal{S}$ and open subset $U \subseteq X$ we define the restriction of the divisorial fan $\mathcal{S}$ to U as $\mathcal{S}|_U := \{\mathcal{D}|_U \mid \mathcal{D} \in \mathcal{S}\}$.

1.4. Appendix one: Varieties as Schemes

In this section we introduce the concept of Scheme, which is a generalization of the concept of Variety. Chapter 2 and 3 can be read with the above background, but the concept of scheme is necessary for a complete understanding. The main observation of this section is that any variety has the structure of a scheme, which was already usted to define Weil divisors on algebraic varieties, and the concept of multiplicity in the inclusion of schemes (see 1.4.6).

Definition 1.4.1 (Locally ringed space). We say that a pair $(X, \mathcal{O}_X)$ is a *ringed space* if X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A morphism of ringed spaces is a continuous map $f : X \to Y$ with a map $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$, where $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$, for each $U \subset Y$ open subset. A ringed space $(X, \mathcal{O}_X)$ is called a *locally ringed space* if for each $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring. A morphism of locally ringed spaces is a morphism of ringes spaces, such that for each $p \in X$, the homomorphism induced in the stalks $f_p^* : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism (the homomorphism f^* is called the *pull-back homomorphism*). An isomorphism of locally ringed spaces is a morphism with a two-sided inverse.

Definition 1.4.2 (Scheme). Given a ring R we denote by $\operatorname{Spec} R$ the set of all prime ideals of R. If $I \subset R$ is a prime ideal, we denote by $Z(I) \subset \operatorname{Spec} R$ the set of all prime ideals which contains I. We call the *Zariski topology* of $\operatorname{Spec} R$ the topology which closed sets have the form $Z(I)$, for each prime ideal $I \subset R$. Also, we can define a sheaf of rings $\mathcal{O}_R$ on $\operatorname{Spec}(R)$ in the following way: For any open subset $U \subset \operatorname{Spec} R$, we define $\mathcal{O}_R(U)$ to be the set of functions $s : U \to \coprod_{p\in U} R_p$, such that $s(p) \in R_p$, for each p, and for each $p \in U$, there exists a neighborhood V of p, contained in U, and elements $a, f \in R$, such that $f \notin q$ for each $q \in V$, and $s(q) = \frac{a}{f}$. In other words, we require s to be locally a quotient of elements of R. We say that a locally ringed space isomorphic to $(R, \mathcal{O}_R)$ is an *affine scheme*. A *scheme* is a locally ringed space which is locally an affine scheme. It means that for each point $p \in X$ of the locally ringed space $(X, \mathcal{O}_X)$, there exists a neighborhood $p \in U$, such that with the restricted sheaf $(U, \mathcal{O}_X|_U)$ is an affine scheme. A morphism and isomorphism of schemes are morphisms and isomorphisms of locally ringed spaces. An affine scheme is said to be *noetherian* if it is the spectrum of a noetherian ring. We say that $\mathcal{O}_X$ is the *structure sheaf* of X. The dimension of a scheme X is the dimension of its topological space. By abuse of notation, we will denote by the same symbol X, a scheme and its topological space. An *irreducible component of a scheme* is an irreducible component of its topological space.

Definition 1.4.3 (Notation of schemes). A scheme X is said to be

- *connected* if its topological space is connected.
- *reduced* if for every open subset $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

- *integral* if for every open subset $U \subset X$, the ring $\mathcal{O}_X(U)$ is an integral domain.
- *locally noetherian* if it can be covered by noetherian affine schemes.
- *noetherian* if it is noetherian and quasi-compact.
- *regular* if for each point $p \in X$, the ring $\mathcal{O}_{p,X}$ is a regular local ring.
- *normal* if for each point $p \in X$, the ring $\mathcal{O}_{p,X}$ is an integrally closed domain.
- *regular in codimension one* if every local ring $\mathcal{O}_{p,X}$ of dimension one is regular. This means that the scheme is regular along the subspace $\overline{\{x\}}$ which has codimension one.

Definition 1.4.4 (Open and closed subschemes). Given an open subset $U \subset X$ of a scheme, we say that U with the structure sheaf $\mathcal{O}_X|_U$ is an *open subscheme* of X. A morphism $f\colon X \to Y$ is an *open immersion* if induces an isomorphism of X with an open subscheme of Y. A *closed immersion* is a morphism $f\colon X \to Y$, of schemes, such that the image of the continuous map between its topological spaces is closed, and the map $f^*\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective. A *closed subscheme* is the image of a closed immersion.

Definition 1.4.5 (Ideal sheaves). Let X be a topological space and $\mathcal{O}_X$ a sheaf of rings on X. An *ideal sheaf* in $\mathcal{O}_X$ is a sheaf $\mathcal{I}$ of $\mathcal{O}_X$-modules, such that $\mathcal{I}(U)$ is an ideal o $\mathcal{O}_X(U)$, for each open subset $U \subset X$.

Given a scheme X, there is a correspondence between closed subschemes of X and quasi-coherent ideal sheaves of the structure sheaf of X. In fact, given an ideal sheaf $\mathcal{I}$ in $\mathcal{O}_X$, then the support of the sheaf $U \mapsto \mathcal{O}_X/\mathcal{I}(U)$ is a closed subspace Y of X, and $(Y, \mathcal{O}_X/J)$ is a closed subscheme of X. On the other hand, the kernel sheaf of the surjective map $f^*\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$, of a closed subscheme $f\colon Y \to X$, gives an ideal sheaf.

Given a scheme X, we define X_{red}, the *unique reduced subscheme of* X to be the subscheme induced by the ideal sheaf which associates the nilradical of $\mathcal{O}_X(U)$ to each open subset $U \subset X$.

The following definition will be used in Chapter 2 to study the subschemes of $\mathrm{Bs}(\mathcal{L})$ which are contained with multiplicity equal or greater than one.

Definition 1.4.6 (Multiplicity along a subscheme). Let X be a scheme and $Z \subset X$ a closed subscheme. Let $Z' \subset Z$ be an irreducible component and $\upsilon \in Z'$ a generic point. Then we define the *multiplicity of* Z' *in* Z to be $\mathrm{length}_{\mathcal{O}_{X,\upsilon}} \mathcal{O}_{Z,\upsilon}$.

Definition 1.4.7 (Fibre product). Let S be a scheme and X,Y schemes with morphisms $X \to S$ and $Y \to S$. The *fibre product* $X \times_S Y$, is a scheme with *projection morphisms* $p_1\colon X \times_S Y \to X$ and $p_2\colon X \times_S Y \to Y$, which makes the square diagram commute, with the following universal property: given another scheme Z with morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ making the square diagram commute, then there exists a unique morphism $h\colon Z \to X \times_S Y$ making the whole diagram commutative. The fibre product of two schemes X and Y with morphisms to a scheme S always exists (see [**26**, Theorem 3.3]).

Definition 1.4.8 (Residue field). Given a scheme X and a point $p \in X$, the *residue field* $\mathbb{K}(p)$ of p on X is the residue field of the local rings $\mathcal{O}_{X,p}$. Given a point $p \in X$, there is a natural morphism $\mathrm{Spec}\,\mathbb{K}(p) \to X$.

Definition 1.4.9 (reduced fibre of a morphism). Given morphism $f\colon X \to Y$, and $y \in Y$ a point. The morphisms f and $\operatorname{Spec}\mathbb{K}(y) \to Y$ induces a fibre product $X_y = X \times_Y \operatorname{Spec}\mathbb{K}(y)$, which is called the *fibre* of f at y. Its topological space is $f^{-1}(y)$. We call $X_{y_{red}}$ the reduced fiber at y. A *generic fiber* of a morphism is a fiber of a generic point.

Definition 1.4.10 (Finite and finite type morphism). A morphism $f\colon X \to Y$ of schemes is said to be *finite* if there exists a coverng of Y by affine subsets $\operatorname{Spec} B_i$, such that for each i, the open set $f^{-1}(\operatorname{Spec} B_i)$ is the spectum of a B_i-algebra which is finitely generated as B_i-module. We say that f is of *finite type* if $f^{-1}(\operatorname{Spec} B_i)$ can be covered by a finite number of affine schemes which are spectrums of finitely generated B_i-algebras.

Definition 1.4.11 (Separated scheme). Given a morphism of schemes $f\colon X \to Y$, the *diagonal morphism of* f is the unique morphism $\Delta_f\colon X \to X \times_Y X$, such that $p_i \circ \Delta_f = id_X$, where $p_1, p_2\colon X \times_Y X \to X$, are the projection on the first and second coordinate respectively. We say that f is a *separated morphism* if Δ_f is a closed immersion. We say that X is separated, if the natural morphism $X \to \operatorname{Spec}\mathbb{Z}$ is separated.

Definition 1.4.12 (Base extension). Let S be a fixed scheme and $S' \to S$ a morphism, then for every scheme X with a morphism $X \to S$ we denote $X' = X \times_S S'$, which is a scheme with a morphism $X' \to S'$. We say that X' is obtained from X via a *base extension* $S' \to S$.

Definition 1.4.13 (Proper morphism). A morphism $f\colon X \to Y$, is *universally closed* if for every morphism $Y' \to Y$, the corresponding morphism obtained by base extension is closed. A morphism is said to be *proper* if it is separated, of finite type, and universally closed.

Definition 1.4.14 (Abstract variety). An *abstract variety* is a integral separated scheme of finite type over an algebraically closed field $\mathbb{K}$. Moreover, we say that the abstract variety is *complete* if it is proper over $\mathbb{K}$.

Remark 1.4.15. The varieties considered in the preliminaries are in fact abstract varieties with its structure sheaf. Since the singular locus of any normal variety has codimension at least two, then a normal variety is in fact a scheme which is regular in codimension one.

1.5. Appendix two: Toric Varietes

In this section we recall the well-known correspondence between toric varieties and fans in finite-dimensional $\mathbb{Q}$-vector spaces. Also, we give a very simple example of a toric variety described as a complexity-one $\mathbb{T}$-variety. Compare the following section with subsection 3.2 to a better understanding of varieties of higher complexity.

Definition 1.5.1 (Fan). Let N be a free finitely generated abelian group, M its dual,$N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ the associated $\mathbb{Q}$-vector spaces. A *fan* Σ in $N_{\mathbb{Q}}$ is a finite collection of pointed convex polyhedral cones such that the intersection of any two elemnts of Σ is a face of both and Σ is closed under taking intersection.

Construction 1.5.2 (Toric variety). Let Σ be a fan in $N_{\mathbb{Q}}$ and $n = \dim_{\mathbb{Q}}(N_{\mathbb{Q}})$. For each cone $\sigma \in \Sigma$, consider the affine variety $X(\sigma) = \operatorname{Spec}(\oplus_{u\in\sigma^\vee}\mathbb{C}\chi^u)$, the grading of the ring $\oplus_{u\in\sigma^\vee}\mathbb{C}\chi^u$, gives $X(\sigma)$ an effective action of $\mathbb{T}=(\mathbb{C}^*)^n$. Given a face $\tau\subset\sigma$, there is a corresponding $\mathbb{T}$-equivariant inclusion $X(\tau)\subset X(\sigma)$, then we can glue equivariantly all the affine varietis $X(\sigma)$, with $\sigma\in\Sigma$, to obtain a $\mathbb{T}$-variety $X(\Sigma)$ of dimension n. There is a one-to-one dimension reversing equivalence between faces of Σ and orbits of $X(\Sigma)$. From the discussion in subsection 3.2 it follows that any toric varieties appears in this way.

Definition 1.5.3 (Notation of toric varieties). A toric variety $X(\Sigma)$ is

- complete if $\cup_{\sigma\in\Sigma}\sigma$, the support of Σ, is all $N_{\mathbb{Q}}$.
- $\mathbb{Q}$-factorial if for each cone $\sigma\in\Sigma$, the extremal rays generating this cone are linearly independent over $\mathbb{Q}$.
- smooth if for each cone $\sigma\in\Sigma$, the extremal rays generating this cone are linearly independent over $\mathbb{Z}$.

Example 1.5.4. Consider the fan Σ of $\mathbb{Q}^2$ whose maximal cones are listed on the left, and its rays are in figure 1.

$$\begin{aligned}\sigma_1 &= \langle(1,0),(0,1)\rangle,\\ \sigma_2 &= \langle(0,1),(-1,-2)\rangle,\\ \sigma_3 &= \langle(-1,-2),(0,-1)\rangle,\\ \sigma_4 &= \langle(0,-1),(1,0)\rangle.\end{aligned}$$

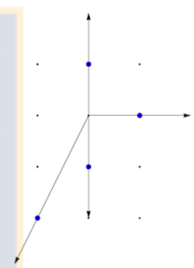


Figure 1.

Observe that $X(\Sigma)$ is an algebraic complete smooth toric surface. Every maximal cone of Σ corresponds to a fixed point of $X(\Sigma)$, each ray of Figure 1 corresponds to a $\mathbb{T}$-invariant divisor of $X(\Sigma)$, and the origin of $\mathbb{Q}^2$ corresponds to $(\mathbb{C}^*)^2$ as an open orbit of $X(\Sigma)$. In fact, $X(\Sigma)$ is the Hirzebruch surface Σ_2 given by

$$\Sigma_2 := \{([x_0:x_1],[y_0:y_1:y_2]) \mid x_0^2y_1 - x_1^2y_2 = 0\}\subset\mathbb{P}^1\times\mathbb{P}^2.$$

Observe that the $(\mathbb{C}^*)^2$ is given by

$$(t_1,t_2)\cdot([x_0:x_1],[y_0:y_1:y_2])\to([t_1x_0:x_1],[t_2y_0:y_1:y_2]).$$

One can consider the subtorus $\mathbb{C}^*=\{(t_1,1)\}\subset(\mathbb{C}^*)^2$, and describe Σ_2 as a complexity-one $\mathbb{T}$-variety with respect to the action of the subtorus. The corresponding divisorial fan $\mathcal{S}_2$ on $(\mathbb{P}^1,\mathbb{Q})$ has four maximal p-divisors given as follows

$$\begin{aligned}\mathcal{D}_1 &= \emptyset\otimes\{0\}+\mathbb{Q}_{\geq 0}\otimes\{\infty\},\\ \mathcal{D}_2 &= [-2,\infty[\otimes\{0\}+\emptyset\otimes\{\infty\},\\ \mathcal{D}_3 &=]-\infty,-2]\otimes\{0\}+\emptyset\otimes\{\infty\},\\ \mathcal{D}_4 &= \emptyset\otimes\{0\}+\mathbb{Q}_{\leq 0}\otimes\{\infty\}.\end{aligned}$$

Since the locus of each p-divisor $\mathcal{D}_i$ is affine, then the contraction morphism $\widetilde{X}(\mathcal{D}_i)\to X(\mathcal{D}_i)$ is an isomorphism. The tail cone of $\mathcal{D}_1$ and $\mathcal{D}_2$ is $\mathbb{Q}_{\geq 0}$ while the tail cone of $\mathcal{D}_3$ and $\mathcal{D}_4$ is $\mathbb{Q}_{\leq 0}$. The tail fan of $\mathcal{S}_2$ is the fan of $\mathbb{P}^1$. The trivial locus of $\mathcal{S}_2$ is $\mathbb{P}^1\setminus\{0,\infty\}$ and the good quotient $\Sigma_2\to\mathbb{P}^1$ has fiber isomorphic to $\mathbb{P}^1$ in the trivial locus. The good quotient of Σ_2 associated to its description as a complexity-one

$\mathbb{T}$-variety is in fact the clasical description of Σ_2 as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Is standard to check that for each $1 \leq i \leq 4$ we have that $X(\mathcal{D}_i) \simeq X(\sigma_i)$ and the gluing construction of Σ_2 as a $\mathbb{T}$-variety or as a toric variety is the same.

CHAPTER 2

Linear Systems on $(\mathbb{P}^1)^n$

2.1. Fiber Special Systems of $(\mathbb{P}^1)^n$

In what follows we will denote by $\mathbb{K}$ an algebraically closed field. Given an algebraic variety X we denote by $h^i(X, D)$ the dimension of the i-th cohomology group of any line bundle whose class is $D \in \mathrm{Pic}(X)$.

In this section we recall some definitions, notations and results about linear systems on $(\mathbb{P}^1)^n$ and $\mathbb{P}^n$. First of all we denote by $\mathbb{K}[x_1, y_1, \ldots, x_n, y_n]$ the Cox ring of $(\mathbb{P}^1)^n$ and by $\mathbb{K}[x_0, \ldots, x_n]$ the Cox ring of $\mathbb{P}^n$. Let $\pi_Y : Y \to (\mathbb{P}^1)^n$ (resp. $\pi_X : X \to \mathbb{P}^n$) be the blowing-up at r (resp. $r+n-1$) points in very general position. The Picard Group of Y is generated by the $r+n$ classes of $H_1, \ldots, H_n, E_1, \ldots E_r$ where E_i is the exceptional divisor over the i-th point and H_i is the pull-back of the prime divisor of equation $x_i = 0$. The Picard Group of X is generated by the $r+n$ classes of $H, E_1, \ldots, E_{r+n-1}$ where E_i is the exceptional divisor over the i-th point and H is the pull-back of a hyperplane. We will call these bases *tautological*.

Remark 2.1.1. (Points in very general position). Let $q_1, \ldots, q_r$ be distinct points of $(\mathbb{P}^1)^n$ and let $m \in \mathbb{N}^r$. Consider the scheme $(\mathbb{P}^1)^n_{[r]}$ parametrizing r-tuples of points in $(\mathbb{P}^1)^n$ and let $\mathcal{Q} \in (\mathbb{P}^1)^n_{[r]}$ be the point $q_1 + \cdots + q_r$. For $d \in \mathbb{N}^n$ denote by $\mathcal{H}(d, m, \mathcal{Q})$ the vector space of degree d homogeneous polynomials of $\mathbb{K}[x_1, y_1, \ldots, x_n, y_n]$ with multiplicity at least m_i at each q_i. Observe that $\mathcal{H}(d, m, \mathcal{Q})$ depends on $\mathcal{Q}$ and that there is a Zariski open subset $\mathcal{U}(d, m) \subset (\mathbb{P}^1)^n_{[r]}$ where this dimension attains its minimal value. Let us denote by

$$\mathcal{U} := \bigcap_{(d,m) \in \mathbb{N}^{n+r}} \mathcal{U}(d, m).$$

We say that the points $q_1, \ldots, q_r$ are in very general position if the corresponding $\mathcal{Q}$ is in $\mathcal{U}$.

Definition 2.1.2. Given a birational map $\phi : X \dashrightarrow Y$ of algebraic varieties we say that ϕ is a *small modification* if there exist open subsets $U \subseteq X$ and $V \subseteq Y$ such that $\varphi(U) \subseteq V$, the restriction $\phi|_U$ is an isomorphism and both $X - U$ and $Y - V$ have codimension at lest two. Note that any small modification induces mutually inverse isomorphisms of push-forward and pull-back

$$\phi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y) \qquad \phi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X).$$

Moreover $h^0(Y, \phi_*(D)) = h^0(X, D)$ for any $D \in \mathrm{Pic}(X)$ and $h^0(X, \phi^*(D)) = h^0(Y, D)$ for any $D \in \mathrm{Pic}(Y)$.

Definition 2.1.3. Let X be an algebraic variety, $D \in \mathrm{Pic}(X)$ a divisor class and $V \subset X$ a subvariety. We say that V is contained with multiplicity m in the base locus of D if the exceptional divisor E of the blow-up $\pi\colon \tilde{X} \to X$ of X at V is contained with multiplicity m in the base locus of $\pi^*(D)$.

REMARK 2.1.4. Let $\phi : \mathbb{P}^n \to (\mathbb{P}^1)^n$ be the birational map defined by $[x_0 : \cdots : x_n] \mapsto ([x_{n-1} : x_n], \ldots, [x_0 : x_n])$. Let $p_1, \ldots, p_{r+n-1}$ be points of $\mathbb{P}^n$ in very general position such that the first $n+1$ are the fundamental ones and let $q_1, \ldots, q_r$ be points of $(\mathbb{P}^1)^n$ such that $q_1 = ([0 : 1], \ldots, [0 : 1])$, $q_2 = ([1 : 0], \ldots, [1 : 0])$ and $q_{i+2} = \phi(p_{i+n+1})$ for $i \in \{1, \ldots, r-2\}$. This gives the following commutative diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ \mathbb{P}^n & \overset{\phi}{\dashrightarrow} & (\mathbb{P}^1)^n, \end{array}$$

where with abuse of notation we are denoting by the same symbol ϕ and its lift. To show that the above lift is a small modification it is enough to consider the case $r = 2$. In this case we have commutative diagrams

$$\begin{array}{ccccc} \Sigma^n_{n+1} & \supseteq & \Sigma & \subseteq & \Sigma^{1,n}_2 \\ \downarrow & & & & \downarrow \\ \Sigma^n & & & & \Sigma^{1,n} \end{array} \qquad \begin{array}{ccccc} X^n_{n+1} & \supseteq & X(\Sigma) & \subseteq & Y^n_2 \\ \downarrow & & & & \downarrow \\ \mathbb{P}^n & & & & (\mathbb{P}^1)^n \end{array}$$

where the first diagram is obtained by completing in two different ways the fan Σ whose cones are exactly the one-dimensional cones of $\mathbb{Z}^n$ generated by the vectors $\{\pm e_1, \ldots, \pm e_n, \pm(e_1 + \cdots + e_n)\}$, while the second diagram is obtained applying the toric functor to the first one. Since the complement of $X(\Sigma)$ in both $X(\Sigma^n_{n+1})$ and $X(\Sigma^n)$ is of codimension at least two, then the corresponding toric birational map $\phi\colon X \dashrightarrow Y$ is small. We recall that the map ϕ and its action on fat points has already been considered in [14].

With the above notation the induced isomorphism $\phi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is given by

$$(2.1.1) \qquad \begin{cases} H & \mapsto \sum_{i=1}^n H_i - (n-1)E_1 \\ E_{n+1} & \mapsto E_2 \\ E_i & \mapsto H_{n+1-i} - E_1 & \text{for } 1 \le i \le n \\ E_i & \mapsto E_{i-n+1} & \text{for } i > n+1. \end{cases}$$

2.1.1 Standard form

Let us recall that we denote by Y the blow-up of $(\mathbb{P}^1)^n$ at r points $q_1, \ldots, q_r$ in very general position. Without loss of generality we can assume the first two points to be $q_1 = ([0 : 1], \ldots, [0 : 1])$, $q_2 = ([1 : 0], \ldots, [1 : 0])$. In this section we recall the definition of a non-degenerate quadratic form $\mathrm{Pic}(Y) \to \mathbb{Z}$ already introduced in [44]. Define the bilinear form $\mathrm{Pic}(Y) \times \mathrm{Pic}(Y) \to \mathbb{Z}$ by $(D_1, D_2) \mapsto D_1 \cdot D_2$ whose values on pairs of elements of the basis $(H_1, \ldots, H_n, E_1, \ldots, E_r)$ are the following:

$$(2.1.2) \qquad H_i \cdot H_j = 1 - \delta_{ij} \qquad E_k \cdot E_s = -\delta_{ks} \qquad H_i \cdot E_k = 0,$$

where $i, j \in \{1, \ldots, n\}$ and $k, s \in \{1, \ldots, r\}$. Observe that the lattice $\mathrm{Pic}(Y)$ equipped with the integer quadratic form induced by the above bilinear form has discriminant group isomorphic to $\mathbb{Z}/(n-1)\mathbb{Z}$ and generated by the class $\frac{1}{n-1}K_Y$. Recall that given a non-degenerate lattice Λ and an element $R \in \Lambda$ with $R^2 = -2$ one can define the *Picard-Lefschetz reflection* defined by R as:

$$\sigma_R \colon \Lambda \to \Lambda \qquad D \mapsto D + (D \cdot R)R.$$

Observe that σ_R is the reflection in Λ with respect to the hyperplane orthogonal to R. The *Weyl group* of Λ, denoted by $W(\Lambda)$ is the subgroup of isometries of Λ generated by the Picard-Lefschetz reflections. For simplicity, given an algebraic variety X and a bilinear form on $\mathrm{Pic}(X)$ we denote the Weyl group of its Picard group by $W(X)$.

The following is a particular case of [**39**, Theorem 1]:

PROPOSITION 2.1.5. *For each transformation* $w\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(Y)$ *of* $W(Y)$, *there is a small modification* $w\colon Y \dashrightarrow Y_w$ *with the following property:* Y_w *is also a blow-up of* $(\mathbb{P}^1)^n$ *in* r *points* $q_1, \dots, q_r$ *in general position and the pull-back of the tautological basis of* $\mathrm{Pic}(Y_w)$ *coincides with the transformation of the tautological basis of* Y *by* w.

In [**44**, Lemma 2.1] a set of generators for $W(Y)$ consists of $n+r-1$ reflections with respect to the following roots:

$$H_1 - E_1 - E_2, \quad H_1 - H_2, \dots, H_{n-1} - H_n, \quad E_1 - E_2, \dots, E_{r-1} - E_r.$$

Let ω be any element of the Weyl group $W(Y)$ and let $\varphi_\omega\colon Y \dashrightarrow Y_\omega$ be the corresponding small modification. We have that φ_ω is the lift of a birational map $\phi_\omega\colon (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n$, moreover Y_ω is the blow-up of $(\mathbb{P}^1)^n$ at points $q'_1, \dots, q'_r$ where $q'_1 = q_1$,$q'_2 = q_2$ and $q'_i = \phi_\omega(q_i)$ for $i \geq 3$. The birational involution of $(\mathbb{P}^1)^n$ associated to the root $H_1 - E_1 - E_2$ is the following [**39**, pag. 128]:

$$([x_1 : y_1], \dots, [x_n : y_n]) \mapsto \left(\left[\frac{1}{x_1} : \frac{1}{y_1}\right], \left[\frac{x_2}{x_1} : \frac{y_2}{y_1}\right], \dots, \left[\frac{x_n}{x_1} : \frac{y_n}{y_1}\right] \right).$$

The birational involution of $(\mathbb{P}^1)^n$ associated to the root $H_i - H_{i+1}$ is the transformations of $(\mathbb{P}^1)^n$ which exchanges the i-coordinate with the $i+1$-coordinate, for $i \in \{1, \dots, n-1\}$. Finally the birational map of $(\mathbb{P}^1)^n$ associated to the root $E_i - E_{i+1}$ is the identity map as we are just relabeling two points between the q_i's.

REMARK 2.1.6. Observe that the map $\phi_*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isometry of lattices.

To see this it is enough to check that ϕ_* preserves the intersection matrix of the basis $(H, E_1, \dots, E_r)$ of $\mathrm{Pic}(X)$. This holds by (2.1.1) and the definition of the bilinear forms on the two lattices [**44**, 2.1]. We recall also that for any $\omega \in W(Y)$ and any $D, D' \in \mathrm{Pic}(Y)$ we have $h^0(Y, D) = h^0(Y_\omega, \omega(D))$ by [**44**, Lemma 2.3] and D is integral if and only if $\omega(D)$ is.

DEFINITION 2.1.7. A class $D = \sum_{i=1}^n d_i H_i - \sum_i^r m_i E_i$ of $\mathrm{Pic}(Y)$ is in *pre-standard form* if the following inequalities hold:

$$d_1 \geq d_2 \geq \dots \geq d_n \geq 0 \qquad m_1 \geq m_2 \geq \dots \geq m_r \qquad \sum_{i=2}^n d_i \geq m_1 + m_2.$$

If in addition $m_r \geq 0$, then D is in *standard form*.

Remark 2.1.8. By Definition 2.1.7 [**37**, Definition 3.1] and the action of ϕ given above we have that a class D in the Picard group of X is in pre-standard form (resp. in standard form) if and only if $\phi_*(D)$ is in pre-standard form (resp. in standard form). In particular by [**37**, Proposition 3.2] we deduce that for any effective class $D \in \mathrm{Pic}(Y)$ there exists a $w \in W(Y)$ such that $w(D)$ is in pre-standard form.

Remark 2.1.9. A (-1)-*class* of $\mathrm{Pic}(Y)$ is the class of an irreducible and reduced divisor E such that $E^2 = E \cdot K = -1$ where $K := \frac{1}{n-1}K_Y$. Observe that this definition coincides with the classical concept of (-1)-class when $n = 2$.

By Remark 2.1.6 and [**37**, Section 4] we conclude the following: The (-1)-classes form an orbit with respect to the action of the Weil group. Moreover if D is a class in standard form, then $w(D) \cdot E \geq 0$ for any (-1)-class E and any $w \in W(Y)$. Finally, some geometric properties of (-1)-curves on surfaces generalize to (-1)-classes: if D is effective and $D \cdot E < 0$ for some (-1)-class E then $E \subset Bs|D|$ and if E, E' are two distinct (-1)-classes having negative product with D then $E \cdot E' = 0$.

The following program given a class $D \in \mathrm{Pic}(Y)$ returns its standard form $D' \in \mathrm{Pic}(Y)$ or returns $0 \in \mathrm{Pic}(Y)$ if the linear system induced by D is empty.

```
Input: (d, m) ∈ N^n × N^r, with r ≥ 2.
Output: (d, m) ∈ N^n × N^r or ∅.
Sort both d = (d_1, ..., d_n) and m = (m_1, ..., m_r) in decreasing order;
while k := Σ_{i=2}^n d_i − m_1 − m_2 < 0 and min(d_1, ..., d_n) ≥ 0 do
    (d_1, m_1, m_2) := (d_1, m_1, m_2) + (k, k, k);
    Sort both d and m in decreasing order;
end
if min(d_1, ..., d_n) < 0 then
    return ∅
else
    return (d, m);
end
```

Algorithm 1: Standard form.

2.1.2 Fiber special systems

Recall that we denote by $\pi \colon Y \to (\mathbb{P}^1)^n$ the blow-up of $(\mathbb{P}^1)^n$ at r points $q_1, \ldots, q_r$ in very general position. Given a subset $I \subseteq \{1, \ldots, n\}$ we denote by $P_I \colon (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{|I|}$ the morphism defined by (if I is empty P_I is the constant morphism to a point)

$$([x_1 : y_1], \ldots, [x_n : y_n]) \quad \mapsto \quad ([x_i : y_i] : i \in I).$$

We denote by $F_{j,I}$ the fiber of P_I through the point q_j for any j. Given a vector $(d_1, \ldots, d_n) \in \mathbb{N}^n$ we will denote by

$$(2.1.3)\; s_I := \sum_{i \in I} d_i \quad \text{and} \quad S_I := 1 + |I| + s_I \quad \text{for any } I \subseteq \{1, \ldots, n\},$$

where $s_\emptyset = 0$ and $S_\emptyset = 1$. Observe that by the assumption made on the points $F_{i,I} \cap F_{j,I} = \emptyset$ for any $i \neq j$. In what follows we use the notation $\mathcal{L} := \mathcal{L}_{(d_1,\dots,d_n)}(m_1,\dots,m_r)$ to denote a general linear system when no confusion arises. We denote by $V(\mathcal{L})$ the subvector space of homogeneous polynomials of $\mathbb{K}[x_1, y_1, \dots, x_n, y_n]$ of degree $(d_1,\dots,d_n)$ and multiplicity at least $m_1,\dots,m_r$ at $q_1,\dots,q_r$ respectively.

Definition 2.1.10. The *virtual dimension* of $\mathcal{L}$ is

$$\mathrm{vdim}(\mathcal{L}) = \prod_{i=1}^{n}(d_i+1) - \sum_{i=1}^{r}\binom{n+m_i-1}{n} - 1$$

the *expected dimension* of $\mathcal{L}$ is $\mathrm{edim}(\mathcal{L}) = \max(\mathrm{vdim}(\mathcal{L}), -1)$. The *fiber dimension* of the linear system $\mathcal{L}$ is

$$\mathrm{fdim}(\mathcal{L}) \quad := \quad \prod_{i=1}^{n}(d_i+1) - \sum_{\substack{1\leq j\leq r \\ I\subseteq\{1,\dots,n\} \\ S_I\leq m_j}} (-1)^{|I|}\binom{m_j - S_I + n}{n} - 1$$

and the the *fiber-expected dimension* is $\mathrm{efdim}(\mathcal{L}) := \max(-1, \mathrm{fdim}(\mathcal{L}))$. We say that $\mathcal{L}$ is *fiber special* if $\dim(\mathcal{L}) > \mathrm{efdim}(\mathcal{L})$ and it is *fiber non-special* otherwise.

Theorem 2.1.11. *For any linear system $\mathcal{L}$ the following inequalities hold* $\dim(\mathcal{L}) \geq \mathrm{efdim}(\mathcal{L}) \geq \mathrm{edim}(\mathcal{L})$.

Proof. Denote by $\Delta(m) \subseteq \mathbb{Z}^n_{\geq 0}$, for $m \geq 1$, the set of integer points of the n-dimensional simplex which is the convex hull of the points: $0, (m-1)e_1, \dots, (m-1)e_n$. Let $V \subseteq \mathbb{K}[x_1,\dots,x_n]$ be the subvector space of polynomials of degree at most $(d_1,\dots,d_n)$. Given $w \in \mathbb{Z}^n_{z\geq 0}$ we define the partial derivative $\partial/\partial x^w$, where $x^w = x_1^{w_1}\cdots x_n^{w_n}$. Let

$$\Phi\colon V \to \mathbb{K}^N \qquad f \mapsto \left(\frac{\partial f}{\partial x^w}(p_j) \,:\, 1 \leq j \leq r \text{ and } w \in \Delta(m_j)\cap\mathbb{Z}^n_{\geq 0}\right)$$

be the function which maps f to the collection of all partial derivatives of f, corresponding to the integer points of the polytope $\Delta(m_j)$, evaluated at p_j for each j. Observe that $\dim(\mathcal{L})$ equals $\dim(\ker(\Phi))$. Moreover if w is an integer vector outside the polytope $\Delta(m_j) \cap \prod_{i=1}^{n}[0, d_i]$, then $\partial f/\partial x^w$ is the zero polynomial. Thus any such w does not impose conditions on the kernel of Φ. Using the inclusion-exclusion principle we see that the number of integer vectors of the polytope $\Delta(m_j) \cap \prod_{i=1}^{n}[0, d_i]$ equals

$$\mu_j = \sum_{\substack{I\subseteq\{1,\dots,n\} \\ m_j\geq S_I}} (-1)^{|I|}\binom{m_j - S_I + n}{n}.$$

Thus the point p_j of multiplicity m_j can impose at most μ_j conditions and the first inequality $\dim(\mathcal{L}) \geq \mathrm{efdim}(\mathcal{L})$ follows. The second inequality follows by observing that the number of integer vectors of $\Delta(m_j)$ is greater than or equal to the number of integer vectors of $\Delta(m_j) \cap \prod_{i=1}^{n}[0, d_i]$. $\square$

REMARK 2.1.12. For a linear system $\mathcal{L}$ of $\mathbb{P}^n$ through multiple base points in very general position, in [8] the authors introduce the liner expected dimension $\operatorname{eldim}(\mathcal{L})$, which takes into account the speciality coming from linear subspaces through some of the points. In that case the authors asset that the inequality $\dim(\mathcal{L}) \geq \operatorname{eldim}(\mathcal{L})$ is equivalent to the weak Fröberg-Iarrobino conjecture [8, Remark 3.4]. The reason why in $(\mathbb{P}^1)^n$ one can easily prove the inequality $\dim(\mathcal{L}) \geq \operatorname{efdim}(\mathcal{L})$ is that the each subvariety taken into account in the fdim formula passes exactly through one point.

THEOREM 2.1.13. *A linear system $\mathcal{L}$ through two points is fiber non-special.*

The proof of the following lemma is a direct consequence of the identity $\sum_{i=n}^{k} \binom{i}{n} = \binom{k+1}{n+1}$ which holds for any $k \geq n$.

LEMMA 2.1.14. *Let I be an ordered subset of $\{1, \ldots, n-1\}$, let $J := I \cup \{n\}$ and let m be a non-negative integer. Given a vector $(d_1, \ldots, d_n) \in \mathbb{N}^n$ let S_I be defined as in (2.1.3). Then the following holds*

$$\sum_{j=0}^{d_n} \sum_{m-j \geq S_I} \binom{m-j-S_I+n-1}{n-1} = \sum_{m \geq S_I} \binom{m-S_I+n}{n} - \sum_{m \geq S_J} \binom{m-S_J+n}{n}.$$

PROOF OF THEOREM 2.1.13. Without loss of generality we can assume that $q_1 := ([0:1], \ldots, [0:1])$, $q_2 := ([1:0], \ldots, [1:0])$. Hence a basis $\mathfrak{B}(\mathcal{L})$ for $V(\mathcal{L})$ consists of the monomials of the form $\prod_{i=1}^{n} x_i^{a_i} y_i^{b_i}$ where $\sum_{i=1}^{n} a_i \geq m_1$, $\sum_{i=1}^{n} b_i \geq m_2$ and $a_i + b_i = d_i$ for any i. The statement follows by induction on n using Lemma 2.1.14 and the equality

$$|\mathfrak{B}(\mathcal{L})| = \sum_{j=0}^{d_n} |\mathfrak{B}(\mathcal{L}_{(d_1,\ldots,d_{n-1})}(m_1 - j, m_2 - d_n + j))|.$$

□

COROLLARY 2.1.15. *A linear system $\mathcal{L} := \mathcal{L}_{(d_1,\ldots,d_n)}(m_1, m_2)$ is effective if and only if $\sum_{i=1}^{n} d_i \geq m_1 + m_2$.*

PROOF. If $\sum_{i=1}^{n} d_i < m_1 + m_2$ then, with the same notation of the proof of Theorem 2.1.13 either $\sum_{i=1}^{n} a_i < m_1$ or $\sum_{i=1}^{n} b_i < m_2$ so that there are no monomials in $V(\mathcal{L})$ and thus $\mathcal{L}$ is empty. On the other hand if $\sum_{i=1}^{n} d_i \geq m_1 + m_2$ then there are a_i, b_i such that $\sum_{i=1}^{n} a_i \geq m_1$ and $\sum_{i=1}^{n} b_i \geq m_2$ and $a_i + b_i = d_i$ for any i. Thus $V(\mathcal{L})$ contains a monomial and hence $\mathcal{L}$ is not empty. □

PROPOSITION 2.1.16. *Let $\mathcal{L}$ be a non-empty linear system. Then the fiber $F_{I,j}$ is contained in the base locus of $\mathcal{L}$ with multiplicity*

$$\mu \geq \max\{m_j - s_{I^c}, 0\}$$

and the equality holds when $r \leq 2$.

PROOF. Without loss of generality we can assume that $j = 1$ and $I = \{1, \ldots i\}$. Let $\mathcal{M} := \mathcal{L}_{(d_1,\ldots,d_n)}(m_1, m_2)$. The vector space $V(\mathcal{L})$ is a subspace of $V(\mathcal{M})$ which admits a monomial basis $\mathcal{B}(\mathcal{M})$ given in the proof of Theorem 2.1.13. The Cox ring of the blow-up $\pi_{1,I} \colon X_{1,I} \to (\mathbb{P}^1)^n$ of $(\mathbb{P}^1)^n$ at $F_{1,I}$ is isomorphic to $\mathbb{K}[zx_1, y_1, \ldots, zx_i, y_i, x_{i+1}, y_{i+1}, \ldots, x_n, y_n]$, where z corresponds to the exceptional divisor. Let $\mathcal{B}'(\mathcal{M})$ be the pull-back of the basis $\mathcal{B}(\mathcal{M})$ via $\pi_{1,I}$. Then the basis $\mathcal{B}'(\mathcal{M})$ consists of the following monomials

$$\prod_{j=1}^{i}(zx_j)^{a_j}(y_j)^{b_j}\prod_{j=i+1}^{n}x_j^{a_j}y_j^{b_j},$$

where $\sum_{i=1}^n a_i \geq m_1, \sum_{i=1}^n b_i \geq m_2$ and $a_i + b_i = d_i$ for each i. Observe that $\sum_{j=1}^i a_j \geq m_1 - \sum_{j=i+1}^n a_j \geq m_1 - s_{I^c}$, with equalities when $b_j = 0$ for any $j \in \{i+1, \ldots, n\}$ and $\sum_{i=1}^n a_i = m_1$. Thus $z^{m_1 - s_{I^c}}$ divides any monomial in $\mathcal{B}'(\mathcal{M})$ and this is the maximal power with this property when $\mathcal{M} = \mathcal{L}$, i.e. when $r \leq 2$. □

2.1.3 Base Locus of the Linear System

In this subsection we describe a class of subvarieties contained in the base locus $\mathrm{Bs}(\mathcal{L})$ of a linear system $\mathcal{L}$ of the form $\mathcal{L}_{(d_1,\ldots,d_n)}(m_1,\ldots,m_r)$ of $(\mathbb{P}^1)^n$ and compute their multiplicity in $\mathrm{Bs}(\mathcal{L})$. By the generality assumption on the position of the points we can assume all but the first two of them to be contained in the n-dimensional torus $\mathbb{T}^n$ of $(\mathbb{P}^1)^n$. Denote by $p_{n+i-1} = \phi^{-1}(q_i)$, for $3 \leq i \leq r$, where $\phi\colon \mathbb{P}^n \to (\mathbb{P}^1)^n$ is the birational map defined in Remark 2.1.4. Given two subsets $I \subseteq \{1,\ldots,n\}$ and $J \subseteq \{2,\ldots,r\}$ we denote by L_{IJ} the following linear subspace of $\mathbb{P}^n$:

$$L_{IJ} = \langle \{e_i : i \in I\} \cup \{p_j : j \in J\}\rangle.$$

We denote by V_{IJ} the closure in $(\mathbb{P}^1)^n$ of $\phi(L_{IJ} \cap \mathbb{T}^n)$. Observe that if L_{IJ} is defined by a matrix $A \in M_{k\times n}(\mathbb{C})$, with $k = |I| + |J| - 1$, then $V_{IJ} \cap \phi(\mathbb{T}^n)$ is defined by the following equations

$$A\begin{bmatrix} y_1y_2\cdots y_{n-1}x_n \\ y_1y_2\cdots x_{n-1}y_n \\ \vdots \\ x_1y_2\cdots y_{n-1}y_n \\ y_1y_2\cdots y_{n-1}y_n \end{bmatrix} = \begin{bmatrix} 0\\0\\ \vdots \\ 0 \\ 0\end{bmatrix}.$$

Proposition 2.1.17. *Let $\mathcal{L} = \mathcal{L}_{(d_1,\ldots,d_n)}(m_1,\ldots,m_r)$ be a non-empty linear system and let V_{IJ} be as above. Then V_{IJ} is contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity*

$$\mu_{IJ} := \max\left\{0, (|J|-1)(m_1-\delta) + \sum_{i\in J} m_i - \sum_{i \in I} d_i\right\},$$

where we denote by $\delta = \sum_{i=1}^n d_i$.

Proof. The multiplicity of V_{IJ} in the base locus of $\mathcal{L}$ equals the multiplicity of L_{IJ} in the base locus of $\phi^*(\mathcal{L})$. By (2.1.1) the class of an element of $\phi^*(\mathcal{L})$ is

$$(\delta - m_1)\,H - \sum_{i=1}^{n}(\delta - m_1 - d_{n-i})\,E_i - \sum_{i=2}^{r} m_{n+i-1}E_{n+i-1}.$$

Thus we conclude by [8, Proposition 2.5]. □

As a consequence of the fact that the base locus of a linear system through $n+2$ points in $\mathbb{P}^n$ is a union of linear subspaces [8, Corollary 4.8] we immediately get the following.

Corollary 2.1.18. *If $\mathcal{L}_{(d_1,\ldots,d_n)}(m_1,m_2,m_3)$ is a linear system through three points in general position, then its base locus only contains varieties of the form $F_{I,j}$ and V_{IJ}.*

2.2. Degeneration of $(\mathbb{P}^1)^n$ and Linear Systems

In this section we use the degeneration in $(\mathbb{P}^1)^n$ introduced in [**34**, Section 3] and using a method introduced in [**21**] we prove a Theorem that allows to check non-speciality of a linear system in $(\mathbb{P}^1)^n$ via this degeneration.

Recall that $\Delta(m+1) \subseteq \mathbb{Z}^n_{\geq 0}$, for $m \geq 0$, the set of integer points of the n-dimensional simplex which is the convex hull of the points: $0, me_1, \ldots, me_n$.

$\mathcal{L}_{(d_1,\ldots,d_{n-1},k\to d_n)}(m_1,\ldots,m_r)$ will denote the sublinear system of $\mathcal{L}$ defined by all the polynomials divisible by x_n^k. We denote by $V_A(\mathcal{L})$ the subvector space of $\mathbb{K}[x_1,\ldots,x_n]$ obtained by evaluating the polynomials of $V(\mathcal{L})$ at $y_1 = \cdots = y_n = 1$. Observe that $V_A(\mathcal{L})$ is the subvector space of polynomials $f \in \mathbb{K}[x_1,\ldots,x_n]$ of degree at most $(d_1,\ldots,d_n)$ such that f has multiplicity at least m_j at p_j for any j. Let $V := V_A(\mathcal{L}_{(d_1,\ldots,d_n)})$ and let

$$\Phi\colon V \to \mathbb{K}^N$$

be the function which maps f into the collection of all partial derivatives of f, which correspond to the integer points of the polytope $\Delta(m_i) \cap \prod_{i=1}^n [0,d_i]$ evaluated at p_i for each i (see also the proof of Theorem 2.1.11). Let $M(\mathcal{L})$ be the matrix of Φ with respect to the monomial basis of V and the standard basis of $\mathbb{K}^N$. The columns $M(\mathcal{L})$ are indexed by monomials of degree at most $(d_1,\ldots,d_n)$, while rows are indexed by conditions imposed by the points. Let $P = \mathbb{K}[p_1^1,\ldots,p_1^n,\ldots,p_r^1,\ldots,p_r^n]$, where p_k^i is the i-coordinate of the k-th point. Then the entries of $M(\mathcal{L})$ can be considered as polynomials in P. Let s be a positive integer $\leq r$, let deg be a grading on P defined by $\deg(p_j^k) = 1$ if $k = n$ and $j \geq s+1$ and $\deg(p_j^k) = 0$ otherwise. In what follows we will adopt the following notation:

$$\mathcal{L}_1 := \mathcal{L}_{(d_1,\ldots,d_{n-1},k-1)}(m_1,\ldots,m_s) \quad \mathcal{L}_2 := \mathcal{L}_{(d_1,\ldots,d_{n-1},d_n-k)}(m_{s+1},\ldots,m_r). \tag{2.2.1}$$

Theorem 2.2.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be defined as in (2.2.1). Assume that the following conditions hold:*

1. *$\mathcal{L}_1$, $\mathcal{L}_2$ are fiber non-special with* $(\mathrm{fdim}(\mathcal{L}_1)+1)(\mathrm{fdim}(\mathcal{L}_2)+1) \geq 0$,
2. *$m_i \leq k$, for any $i \in \{1,\ldots,s\}$,*
3. *$m_j \leq d_n - k + 1$ for any $j \in \{s+1,\ldots,n\}$.*

Then the system $\mathcal{L} := \mathcal{L}_{(d_1,\ldots,d_n)}(m_1,\ldots,m_r)$ is fiber non-special.

Proof. Observe that we have an isomorphism of vector spaces

$$\Psi\colon V_A(\mathcal{L}_2) \to V_A(\mathcal{L}_{(d_1,\ldots,d_{n-1},k\to d_n)}(m_{s+1},\ldots,m_r))$$

where the multiplicities are imposed at the points $p_{s+1},\ldots,p_r$ respectively. After reordering the rows and the columns of the matrix $M(\mathcal{L})$ we can assume that its first γ columns are indexed by monomials of degree at most $(d_1,\ldots,d_{n-1},k-1)$ and that its first ρ rows are indexed by conditions imposed at the points $p_1,\ldots,p_s$. We write

$$M(\mathcal{L}) = \begin{bmatrix} M_1 & K_1 \\ K_2 & M_2 \end{bmatrix},$$

where M_1 is a $\rho \times \gamma$ matrix. Observe that $M_1 = M(\mathcal{L}_1)$ and $M_2 \cong M(\mathcal{L}_2)$ via the isomorphism Ψ. Moreover by conditions (2), (3) and the fact that $\mathcal{L}_1$, $\mathcal{L}_2$ are fiber non-special, we deduce that both matrices have maximal rank. Assume now that $\mathrm{fdim}(\mathcal{L}_1) \geq -1$ and $\mathrm{fdim}(\mathcal{L}_2) \geq -1$ (the other case being analysed in a similar way). Choose two submatrices M_i' of M_i of maximal rank, for $i \in \{1,2\}$, and form the square submatrix of $M(\mathcal{L})$

$$M' = \begin{bmatrix} M_1' & K_1' \\ K_2' & M_2' \end{bmatrix}$$

where K_1' is obtained from K_1 by deleting columns of M_2 and similarly for K_2'. By [**21**, Lemma 2] we have that $\deg(\det(M_2')) > \deg(\det(B))$ for any square submatrix B of $[K_2' \;\; M_2']$. Thus, by the Laplace expansion with respect to the first ρ rows we conclude that $\deg(\det(M')) = \deg(\det(M_1') \cdot \det(M_2')) > 0$ and the result follows. □

The following algorithm is a recursive program that uses Theorem 2.2.1 and Theorem 2.1.13 in order to conclude if the given linear system is non-special.

Input: $(d,m) \in \mathbb{N}^n \times \mathbb{N}^r$, with $r \geq 2$.
Output: $x \in \{$non-special, undecided, special$\}$.
if $\mathrm{std}(d,m) = \emptyset$ **then**
  return non-special.
else if $\mathrm{fdim}(\mathrm{std}(d,m)) > \mathrm{edim}(d,m)$ **then**
  return special;
else
  $(d,m) := \mathrm{std}(d,m)$;
  if $r = 2$ **then**
    if $\mathrm{fdim}(d,m) \geq \mathrm{edim}(d,m)$ **then**
      return special;
    else
      return non-special;
    end
  else
    for $k \in \{1,\dots,d_1-1\}$, $s \in \{1,\dots,r-1\}$ **do**
      $d' := (k-1, d_2, \dots, d_n)$, $m' := (m_1,\dots,m_s)$;
      $d'' := (d_1-k, d_2,\dots,d_n)$, $m'' := (m_{s+1},\dots,m_r)$;
      if $\mathrm{sp}(d',m')$ =non-special *and* $\mathrm{sp}(d'',m'')$ =non-special
      and $(\mathrm{fdim}(d',m')+1)(\mathrm{fdim}(d'',m'')+1) \geq 0$
      and $m_i \leq k$ *for any* $i \in \{1,\dots,s\}$
      and $m_j \leq d_n - k$ *for any* $j \in \{s+1,\dots,r\}$
      then
        return non-special;
      end
    end
    return undecided;
  end
end

Algorithm 2: Speciality by degeneration.

2.2.1 Examples and Conclusions

We have studied linear systems of $(\mathbb{P}^1)^n$ passing through points in very general position and concluded that the fibers of the projections $(\mathbb{P}^1)^n \to (\mathbb{P}^1)^k$, for $1 \le k < n$, can contribute to the speciality. The following is an example of a fiber special linear system $\mathcal{L}$ whose standard form $\mathcal{L}'$ is fiber non-special.

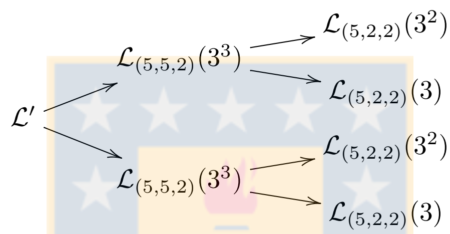
Example 2.2.2. The linear system $\mathcal{L} := \mathcal{L}_{(13,9,5)}(11^2, 7^2, 3^2)$ of $(\mathbb{P}^1)^3$ is not in standard form with

$$\operatorname{vdim}(\mathcal{L}) = 12^2 \cdot 8^2 \cdot 4^2 - 2\left(\binom{13}{3} + \binom{9}{3} + \binom{5}{3}\right) = 80 \qquad \operatorname{fdim}(\mathcal{L}) = 154.$$

Using Algorithm 1 we obtain the following linear systems

$$\mathcal{L}_{(13,9,5)}(11^2, 7^2, 3^2) \rightsquigarrow \mathcal{L}_{(5,9,5)}(7^2, 3^4) \rightsquigarrow \mathcal{L}_{(5,5,5)}(3^6) =: \mathcal{L}'$$

where $\mathcal{L}'$ is in standard form. Algorithm 2 degenerates $\mathcal{L}'$ according to the following scheme:



By Theorem 2.1.13 the last four linear systems are non-special, thus by repeated applications of Theorem 2.2.1 we conclude that $\mathcal{L}'$ is non-special as well. In particular $\dim(\mathcal{L}) = \dim(\mathcal{L}') = \operatorname{vdim}(\mathcal{L}') = 156$.

The following example shows that there are other varieties giving contribution to the speciality of the linear system already when we blow-up three points in very general position.

Example 2.2.3. The linear system $\mathcal{L} := \mathcal{L}_{(1,1,1,1,1,1,1)}(3^3)$ of $(\mathbb{P}^1)^7$ is in standard form with

$$\operatorname{vdim}(\mathcal{L}) = 2^7 - 3\binom{9}{7} = 20 \qquad \operatorname{fdim}(\mathcal{L}) = \operatorname{vdim}(\mathcal{L}) + 21 = 41,$$

where the contribution on the right is given by the 21 one-dimensional fibers on the base locus, but we have that $\dim(\mathcal{L}) = 42$, then $\mathcal{L}$ is fiber-special. Observe that Algorithm 2 returns *undecided* in this case since every degeneration gives a special linear system. The dimension of $\mathcal{L}$ can be calculated by evaluating directly the rank of the matrix $M(\mathcal{L})$, appearing in the proof of Theorem 2.2.1. By Corollary 2.1.18 the base locus of $\mathcal{L}$ is the union of all the fibers through each of the three points plus the irreducible surfaces V_{IJ} for $J = \{2,3\}$ and $I = \{i\} \subseteq \{1, \dots, 7\}$, plus the curve $C = V_{\emptyset,\{2,3\}}$. By Proposition 2.1.17 each V_{IJ} is contained in the base locus of $\mathcal{L}$ with multiplicity 1 and C is contained with multiplicity 2. Moreover the equality

$$\dim(\mathcal{L}) = \operatorname{fdim}(\mathcal{L}) + 1$$

suggests that C is contributing to the speciality of $\mathcal{L}$.

Remark 2.2.4. Observe that the strict inequality $\dim(\mathcal{L}) > \operatorname{efdim}(\mathcal{L})$ can hold also in the simple case when all the multiplicities equal 2. For instance the linear system $\mathcal{L} = \mathcal{L}_{(2,2,2)}(2^7)$ is special of dimension 0 and $\operatorname{efdim}(\mathcal{L}) = -1$. The subvariety of $(\mathbb{P}^1)^3$ which produces the speciality is the unique surface of the linear system $\mathcal{L}_{(1,1,1)}(1^7)$. For a complete classification of the base loci of special linear systems through double points of $(\mathbb{P}^1)^n$ see [**34**, Section 7].

Denote, as before, by Y the blow-up of $(\mathbb{P}^1)^3$ at r points in very general position and by $\phi\colon \mathbb{P}^3 \to (\mathbb{P}^1)^3$ the birational map defined in Remark 2.1.4. Let Q be a divisor in the strict transform of the linear system $\mathcal{L}_{(1,1,1)}(1^7)$ which is the image via ϕ^* of the class of the strict transform of the quadric through 9 points of $\mathbb{P}^3$. For any divisor D in the strict transform of $\mathcal{L}_{(d_1,d_2,d_3)}(m_1,\ldots,m_7)$ let

$$q(D) := \chi(D|_Q) = (d_1+1)(d_2+1)(d_3+1) - d_1d_2d_3 - \sum_{i=1}^{7} \frac{m_i(m_i+1)}{2}.$$

The following conjecture is equivalent to [**37**, Conjecture 6.3] via the small modification ϕ.

Conjecture 2.2.5. *Let $\mathcal{L} := \mathcal{L}_{(d_1,d_2,d_3)}(m_1,\ldots,m_r)$ be a linear system in standard form and let D be a divisor in its strict transform.*

- *If $q(D) \leq 0$, then $h^0(D) = h^0(D-Q)$.*
- *If $q(D) > 0$, then D is special if and only if $m_1 > d_n + 1$ and D is fiber non-special.*

Example 2.2.6. Let $\mathcal{L}_n = \mathcal{L}_{(n,n,n)}(n^7)$ be the linear system corresponding to the divisor class $nQ \in \mathrm{Pic}(Y)$, where $n > 0$. This system has dimension 1 for any n and it is non-special for $n = 1$. Its fiber dimension is

$$\mathrm{fdim}(\mathcal{L}_n) = \mathrm{vdim}(\mathcal{L}_n) = n^3 - 7\binom{2n-1}{n} < 0,$$

so that $\mathcal{L}_n$ is fiber-special for $n > 1$. It is easy to check that $q(nQ) = 0$ for any n and that in this case the conjecture holds.

CHAPTER 3

Topology of Complexity-one $\mathbb{T}$-Varieties

3.1. Fundamental group of $\mathbb{T}$-varieties

This section is devoted to prove Theorem 3 in the introduction. Let us first remark that the log-terminal assumption in Theorem 3 is essential. Indeed, let $\mathcal{D}$ be any p-divisor on a projective curve Y with positive genus, then by [**35**, Corollary 5.4] the $\mathbb{T}$-variety $X := X(\mathcal{D})$ is not log-terminal. Moreover, since $Y = \mathrm{Loc}(\mathcal{D})$ is projective, X has an attractive fixed point and thus $\pi_1(X)$ is trivial, while $\pi_1(\mathrm{Loc}(D))$ is non-trivial.

Given a p-divisor $\mathcal{D}$ on a normal projective variety Y, we denote by $(N_{\mathcal{D}})_{\mathbb{Q}}$ the subspace of $N_{\mathbb{Q}}$ generated by the subset

$$\{v_1 - v_2 \mid y \in Y, v_1, v_2 \in \mathcal{D}_y\}.$$

We denote by $N_{\mathcal{D}} := N \cap (N_{\mathcal{D}})_{\mathbb{Q}}$ the sublattice of N of the integer points of $(N_{\mathcal{D}})_{\mathbb{Q}}$ and by $N(\mathcal{D})$ the quotient $N/N_{\mathcal{D}}$. Given a divisorial fan $\mathcal{S}$ on a variety Y, we denote by $N_{\mathcal{S}}$ the sublattice of N generated by $\{N_{\mathcal{D}} \mid \mathcal{D} \in \mathcal{S}\}$ and by $N(\mathcal{S})$ the following quotient

$$N(\mathcal{S}) := N/N_{\mathcal{S}}.$$

Definition 3.1.1 (See [**30**]). We say that a divisorial fan $\mathcal{S}$ on $(Y, N_{\mathbb{Q}})$ is *contraction-free* if for all $\mathcal{D} \in \mathcal{S}$ the locus of $\mathcal{D}$ is affine.

Observe that $\mathcal{S}$ is contraction-free if and only if for every $\mathcal{D} \in \mathcal{S}$ the contraction morphism $\widetilde{X}(\mathcal{D}) \to X(\mathcal{D})$ is an isomorphism. Recall that for any divisorial fan S we have a rational quotient map $\pi\colon X(\mathcal{S}) \dashrightarrow \mathrm{Loc}(\mathcal{S})$ and if $\mathcal{S}$ is contraction-free observe that this rational quotient map is indeed a morphism $\pi\colon X(\mathcal{S}) \to \mathrm{Loc}(\mathcal{S})$. Given a contraction-free divisorial fan $\mathcal{S}$ on $(Y, N_{\mathbb{Q}})$ and a open subset $V \subseteq Y$, we will adopt the following notation

$$U := \pi^{-1}(V) \qquad V_t := V \cap V_{\mathcal{S}} \qquad U_t := \pi^{-1}(V_t).$$

In order to prove Theorem 3, we will prove Proposition 3.1.2 via relaxing the hypothesis successively in Lemmas 3.1.3, 3.1.4, 3.1.6 and 3.1.7. In what follows we will assume that Y is a smooth variety and that all the divisors of the following four lemmas have tail cone $\sigma(\mathcal{D}) = \sigma$. In order to abbreviate the notation we will assume that V is contained in $\mathrm{Loc}(\mathcal{D})$ in the following lemmas.

Proposition 3.1.2. *Let $\mathcal{S}$ be a contraction-free divisorial fan on $(Y, N_{\mathbb{Q}})$ such that $X(\mathcal{S})$ is smooth and $V \subseteq Y$ an open subset. Then the trivialization $t\colon U_t \simeq$*

$T_N \times V_t$ induces the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \pi_1(U_t) & \longrightarrow & \pi_1(U) & \longrightarrow & 0 \\
 & & \downarrow \simeq & & t^* \downarrow \simeq & & \downarrow \simeq & & \\
0 & \longrightarrow & K_{\mathcal{S}} \times K_V & \longrightarrow & N(\Sigma(\mathcal{S})) \times \pi_1(V_t) & \xrightarrow{\alpha \times \beta} & N(\mathcal{S}|_V) \times \pi_1(V) & \longrightarrow & 0
\end{array}$$

where $\alpha \colon N(\Sigma(\mathcal{S})) \to N(\mathcal{S}|_V)$ is induced by the inclusion $N(\mathcal{S}|_V) \times \pi_1(V)$ and $\beta \colon \pi_1(V_t) \to \pi_1(V)$ is induced by the inclusion $V_t \to V$.

Recall that given a p-divisor $\mathcal{D}$ on a variety Y we have a good quotient $\pi \colon \widetilde{X}(\mathcal{D}) \to Y$, that for any open subset $V \subseteq Y$ we have an isomorphism $U_t \simeq X(\sigma(\mathcal{D})) \times V_t$ and given a point $y \in V_t$ we have a commutative diagram

$$\begin{array}{ccccc}
T_N & \xrightarrow{t \mapsto (t,v)} & T_N \times V_t & \longrightarrow & V_t \\
\downarrow & & \downarrow & & \downarrow \\
X(\sigma(\mathcal{D})) & \longrightarrow & \widetilde{X} & \longrightarrow & Y
\end{array}$$

where the vertical arrows are open embeddings and $X(\sigma(\mathcal{D})) \to \widetilde{X}$ is the inclusion of the fiber $\pi^{-1}(y) = X(\sigma(\mathcal{D}))$ over y. Passing to the fundamental group and using [**16**, Theorem 12.1.10] we have the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & N \times \pi_1(V_t) & \longrightarrow & \pi_1(V_t) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & N(\sigma(\mathcal{D})) & \longrightarrow & \pi_1(\tilde{X}) & \longrightarrow & \pi_1(Y) & \longrightarrow & 1
\end{array} \tag{3.1.1}$$

If $\widetilde{X}$ and Y are smooth, by [**40**, Lemma 1.5.C] the rows are exact and by [**16**, Theorem 12.1.5] the vertical arrows are surjective.

Lemma 3.1.3. *Let $\mathcal{D}$ be a p-divisor on $(Y, N_{\mathbb{Q}})$ such that X is smooth. If there is only one maximal intersection set, and $N(\mathcal{D})$ is trivial, then Proposition 3.1.2 holds for $V = Y$ affine.*

Proof. Y is affine so we have $\widetilde{X} = X$. Since there is only one maximal intersecting set there exists a point $y \in Y$ in the intersection of all divisors of $\operatorname{supp}(\mathcal{D})$. Moreover, since $N(\mathcal{D})$ is trivial, the polyhedron $\mathcal{D}_y$ is full dimensional in $N_{\mathbb{Q}}$. By [**3**, Proposition 7.6] we have that the fiber $\pi^{-1}(y)$ of the good quotient $\pi \colon \widetilde{X} \to Y$, contains a fixed point p, with respect to the torus action. Let $p_0 \in \widetilde{X}$ be a point on an irreducible fiber $\pi^{-1}(y_0)$ of π. By [**16**, Theorem 12.1.5] the inclusion $T_N \to \pi^{-1}(y_0)$ induces a surjective homomorphism $i^* \colon \pi_1(T_N) \to \pi_1(\pi^{-1}(y_0))$, so that any loop η in the fiber is homotopically equivalent to a loop in T_N. In particular, we can assume without loss of generality that $\eta(t) = \alpha(t) \cdot q$, where α is a loop in T_N with base point the identity. Let γ be a path from q_0 to p. The homotopy $H \colon I^2 \to \widetilde{X}$, defined by $(t, s) \mapsto \alpha(t) \cdot \gamma(s)$, contracts the loop α to the constant loop at p. Thus the homomorphism $N \to \pi_1(\widetilde{X})$ is trivial and the first statement of Proposition 3.1.2 holds. The second statement follows from the commutative diagram (3.1.1). ☐

Lemma 3.1.4. *Let $\mathcal{D}$ be a p-divisor on $(Y, N_{\mathbb{Q}})$ such that X is smooth. If there is only one maximal intersection set, any divisor of* $\operatorname{supp}(\mathcal{D})$ *is principal and $\mathcal{D}$ have at least one polyhedral coefficient of positive dimension, then Proposition 3.1.2 holds for $V = Y$ affine.*

Proof. Write $\mathcal{D} = \sum_D \Delta_D \otimes D$. For each D let $v_D \in N$ such that $\Delta_D - v_D \subseteq (N_{\mathcal{D}})_{\mathbb{Q}}$ and let $\mathcal{D}' := \sum_D (\Delta_D - v_D) \otimes D$. By hypothesis the divisor $\mathcal{D} - \mathcal{D}'$ is principal and thus the sheaves of algebras $\mathcal{A}(\mathcal{D})$ and $\mathcal{A}(\mathcal{D}')$ are isomorphic. Being $\mathcal{D}'$ contained in a proper subspace $(N_{\mathcal{D}})_{\mathbb{Q}}$ of $N_{\mathbb{Q}}$ we conclude that $\tilde{X} = \operatorname{Spec}_{\operatorname{Loc}(\mathcal{D})} \mathcal{A}(\mathcal{D}) \simeq \operatorname{Spec}_{\operatorname{Loc}(\mathcal{D})} \mathcal{A}(\mathcal{D}')$ is isomorphic to the cartesian product $T_{N(\mathcal{D})} \times \operatorname{Spec}_{\operatorname{Loc}(\mathcal{D})} \mathcal{A}(\mathcal{D}'_0)$, where $\mathcal{D}'_0$ is the polyhedral divisor $\mathcal{D}'$ whose coefficients lie in $(N_{\mathcal{D}})_{\mathbb{Q}}$. Then, the first statement follows by Lemma 3.1.3. The second statement follows by the commutative diagram (3.1.1). □

Remark 3.1.5. Suppose that all the polyhedral coefficients of $\mathcal{D}$ are points and write $\mathcal{D} = \sum_D q_D \otimes D$. Let L be the line spaned by two polyhedral coefficients of $\mathcal{D}$. For any point q_D choose $v_D \in N$ such that $q_D - v_D \in L$. Let $l \in L \cap N$ and denote by $\mathcal{D}' := \sum_D (q_D - v_D - l) \otimes D$. As in the previous paragraph we can write $\widetilde{X} \simeq T_{N(l)} \times \operatorname{Spec}_{\operatorname{Loc}(\mathcal{D})} \mathcal{A}(\mathcal{D}'_0)$, where $\mathcal{D}'_0$ is the polyhedral divisor $\mathcal{D}'$ whose coefficients lies in $N_l \simeq \mathbb{Q}$.

Lemma 3.1.6. *Let $\mathcal{D}$ be a p-divisor on $(Y, N_{\mathbb{Q}})$ such that X is smooth. If there is only one maximal intersection set and any divisor of* $\operatorname{supp}(\mathcal{D})$ *is principal, then Proposition 3.1.2 holds for $V = Y$ affine.*

Proof. By Lemma 3.1.4 we reduce to the case when all the polyhedral coefficients of $\mathcal{D}$ are points and using remark 3.1.5 we can reduce to the case $N \simeq \mathbb{Z}$. Observe that in this case the morphism $X = \tilde{X} \to Y$ is a topological $\mathbb{C}^*$-fibration. Let $m \in \mathbb{Z}_{>0}$ such that all the points of $m\mathcal{D}$ are integrals, then we have that $X(m\mathcal{D}) \simeq \mathbb{C}^* \times Y$. Observe that the morphism $X(\mathcal{D}) \to X(m\mathcal{D})$ is the quotient by a cyclic group of order m. Both p-divisors have the same trivial open subset and we have the following commutative diagram

$$\begin{array}{ccc}
\mathbb{C}^* \times V_{\mathcal{D}} & \xleftarrow[(z^m,u) \leftarrow\!\shortmid (z,u)]{} & \mathbb{C}^* \times V_{\mathcal{D}} \\
\downarrow & & \downarrow \\
\mathbb{C}^* \times Y & \longleftarrow & X(\mathcal{D})
\end{array}$$

where the vertical arrows are open embeddings. Passing to the fundamental group we have the following commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} \times \pi_1(V_{\mathcal{D}}) & \xleftarrow[(mz,u) \leftarrow\!\shortmid (z,u)]{} & \mathbb{Z} \times \pi_1(V_{\mathcal{D}}) \\
\downarrow & & \downarrow \\
\mathbb{Z} \times \pi_1(Y) & \longleftarrow & \pi_1(X(\mathcal{D}))
\end{array}$$

The vertical arrows are surjective morphisms by [**16**, Theorem 12.1.5]. By the commutativity of the diagram we conclude that $\pi_1(X(\mathcal{D})) \simeq \mathbb{Z} \times (\pi_1(V_{\mathcal{D}})/K)$, where K is a normal subgroup of $\pi_1(V_{\mathcal{D}})$. On the other hand, we have the long exact sequence of the fibration $X(\mathcal{D}) \to Y$:

$$
\begin{array}{ccccccc}
\pi_1(\mathbb{C}^*) & \longrightarrow & \pi_1(X(\mathcal{D})) & \longrightarrow & \pi_1(Y) & & \\
\downarrow \simeq & & \downarrow \simeq & & \| & & \\
\mathbb{Z} & \longrightarrow & \mathbb{Z} \times (\pi_1(V_{\mathcal{D}})/K) & \longrightarrow & \pi_1(Y) & \longrightarrow & 0.
\end{array}
$$

Choosing $\mathbb{C}^*$ to be a fiber of a point in the trivial open subset of Y we can see that the homomorphism $\mathbb{Z} \to \mathbb{Z} \times (\pi_1(V_{\mathcal{D}})/K)$ is the inclusion on the first component. Thus $\pi_1(X(\mathcal{D})) \simeq \mathbb{Z} \times \pi_1(Y)$ and the result follows. $\square$

LEMMA 3.1.7. *Let $\mathcal{D}$ be a p-divisor on $(Y, N_{\mathbb{Q}})$ such that X is smooth. Then Proposition 3.1.2 holds.*

PROOF. We split the proof in four steps.

(1) Observe that if Proposition 3.1.2 holds for the open sets $V^1 \subseteq V^2 \subseteq Y$ then we have a commutative diagram

$$
\begin{array}{ccc}
N(\sigma(\mathcal{D})) \times \pi_1(V_t^1) & \xrightarrow{\mathrm{id} \times i_{12}^*} & N(\sigma(\mathcal{D})) \times \pi_1(V_t^2) \\
\downarrow \alpha^1 \times \beta^1 & & \downarrow \alpha^2 \times \beta^2 \\
N(\mathcal{D}|_{V^1}) \times \pi_1(V^1) & \longrightarrow & N(\mathcal{D}|_{V^2}) \times \pi_1(V^2)
\end{array}
$$

where $i_{12} \colon V^1 \to V^2$ is the inclusion. We conclude that the bottom horizontal map is of the form $s_{12} \times i_{12}^*$, where s_{12} is the surjection induced by $N_{\mathcal{D}|_{V_2}} \to N_{\mathcal{D}|_{V_1}}$.

(2) If Proposition 3.1.2 holds for the open sets $V^1, V^2 \subseteq Y$ and $V^1 \cap V^2$ then by Seifert Van-Kampen Theorem [**27**, Theorem 1.20] we have two push-out diagrams

$$
\begin{array}{ccc}
N(\mathcal{D}|_{V^1 \cap V^2}) & \longrightarrow & N(\mathcal{D}|_{V^1}) \\
\downarrow & & \downarrow \\
N(\mathcal{D}|_{V^2}) & \longrightarrow & N(\mathcal{D}|_{V^1 \cup V^2})
\end{array}
\qquad
\begin{array}{ccc}
\pi_1(V^1 \cap V^2) & \longrightarrow & \pi_1(V^1) \\
\downarrow & & \downarrow \\
\pi_1(V^2) & \longrightarrow & \pi_1(V^1 \cup V^2)
\end{array}
$$

Taking the free product of the two diagrams and using Step (1) we get the following push-out diagram

$$
\begin{array}{ccc}
\pi_1(U^1 \cap U^2) & \xrightarrow{\imath_1} & \pi_1(U^1) \\
\downarrow \imath_2 & & \downarrow \\
\pi_1(U^2) & \longrightarrow & N(\mathcal{D}|_{V^1 \cup V^2}) \times \pi_1(V^1 \cup V^2)
\end{array}
$$

where the maps $\imath_1$ and $\imath_2$ are induced by the inclusion of the corresponding open subsets. Then $\pi_1(U_1 \cup U_2)$ is isomorphic to $N(\mathcal{D}|_{V^1 \cup V_2}) \times \pi_1(V^1 \cup V^2)$.

Moreover, observe that we have a commutative diagram

$$\begin{array}{ccccc}
\pi_1(U_t^1 \cap U_t^2) & \longrightarrow & \pi_1(U_t^1) & & \\
\downarrow & & \downarrow & \searrow^{p_1} & \\
\pi_1(U_t^2) & \longrightarrow & \pi_1(U_t^1 \cup U_t^2) & & \\
& \searrow_{p_2} & & \overset{p}{\dashrightarrow} & \pi_1(U^1 \cup U^2)
\end{array}$$

where p_i is the composition $\pi_1(U_t^i) \to \pi_1(U^i) \to \pi_1(U^1 \cup U^2)$ induced by the two inclusion maps and the square diagram is a push-out. Then by universal property of the push-out diagram the unique homomorphism p is induced by the inclusion $U_t^1 \cup U_t^2 \subseteq U^1 \cup U^2$. Moreover we have a commutative diagram

$$\begin{array}{ccc}
\pi_1(U_t^1 \cup U_t^2) & \xrightarrow{\quad p \quad} & \pi_1(U^1 \cup U^2) \\
\downarrow \simeq & & \downarrow \simeq \\
N(\sigma(\mathcal{D})) \times \pi_1(V_t^1 \cup V_t^2) & \xrightarrow{\ p' \ } & N(\mathcal{D}|_{V^1 \cup V^2}) \times \pi_1(V^1 \cup V^2)
\end{array}$$

where p' is induced by the inclusions $N_{\sigma(\mathcal{D})} \subseteq N_{\mathcal{D}|_{V^1 \cup V^2}}$ and $V_t^1 \cup V_t^2 \subseteq V^1 \cup V^2$. Thus, Proposition 3.1.2 holds for $V^1 \cup V^2$.

(3) Suppose that Proposition 3.1.2 holds for the open subsets $V_1, \ldots, V_k$ of Y and for every intersection of this sets. We conclude by Step (2) that Proposition 3.1.2 also holds for $\cup_{i=1}^k V_i$. Indeed, suppose by induction that it holds for $\cup_{i=1}^{k-1} V_i$ and $V_k \cap (\cup_{i=1}^{k-1} V_i) = \cup_{i=1}^{k-1} (V_k \cap V_i)$, then by Step (2) it also holds for $\cup_{i=1}^k V_i$.

(4) Let $V \subseteq Y$ be any open subset. Consider a finite affine open cover V_i of V such that every divisor of $\mathrm{supp}(\mathcal{D})$ is principal at any V_i, and $\mathcal{D}|_{V_i}$ has only one maximal intersection set. Let V' be any finite intersection of the V_i's, then V' is an open affine set, so $\mathcal{D}|_{V'}$ is a p-divisor on V' with only one maximal intersection set. By Lemma 3.1.6 we conclude that Proposition 3.1.2 holds for V'. Thus, we are in situation of Step (3) and Proposition 3.1.2 also holds for V.

□

Proof of Proposition 3.1.2. First we prove the Theorem in the case $V = Y$. We proceed by induction on the number n of p-divisors of $\mathcal{S}$. If $n = 1$ then $X(\mathcal{S}) = X(\mathcal{D})$ and results follows from Lemma 3.1.7. Suppose that $\mathcal{S}$ is the set of p-divisors $\{\mathcal{D}^1, \ldots, \mathcal{D}^n\}$ and the result is true for $n-1$. Assume that $\mathcal{D}^n$ is maximal with respect to the inclusion. Using the induction hypothesis on the divisorial fans

$$\mathcal{S}_1 := \{\mathcal{D}^n\}, \quad \mathcal{S}_2 := \{\mathcal{D}^1, \ldots, \mathcal{D}^{n-1}\}, \quad \mathcal{S}' := \{\mathcal{D}^1 \cap \mathcal{D}^n, \ldots, \mathcal{D}^{n-1} \cap \mathcal{D}^n\}.$$

By a similar argument as in the proof of Lemma 3.1.7 we have a commutative diagram

$$\begin{CD} N(\sigma(\mathcal{S}')) \times \pi_1(V_{\mathcal{S}'}) @>>> N(\sigma(S_i)) \times \pi_1(V_{\mathcal{S}_i}) \\ @VVV @VVV \\ N(\mathcal{S}') \times \pi_1(\mathrm{Loc}(\mathcal{S}')) @>{\alpha_i \times \beta_i}>> N(S_i) \times \pi_1(\mathrm{Loc}(\mathcal{S}_i)), \end{CD}$$

for each $i \in \{1,2\}$. Thus the homomorphisms induced by the inclusion $X(\mathcal{S}') \to X(\mathcal{S}_i)$ is $\alpha_i \times \beta_i$ where on each factor the corresponding homomorphism is induced by the inclusions $\mathrm{Loc}(\mathcal{S}) \to \mathrm{Loc}(\mathcal{S}_i)$ and $N_{\mathcal{S}_i} \to N_{\mathcal{S}}$, for each i. We have the following push-out diagram

$$\begin{CD} N(\mathcal{S}') \times \pi_1(\mathrm{Loc}(\mathcal{S}')) @>{\alpha_1 \times \beta_1}>> N(\mathcal{S}_1) \times \pi_1(\mathrm{Loc}(\mathcal{S}_1)) \\ @V{\alpha_2 \times \beta_2}VV @VVV \\ N(\mathcal{S}_2) \times \pi_1(\mathrm{Loc}(\mathcal{S}_2)) @>>> N(\mathcal{S}) \times \pi_1(\mathrm{Loc}(\mathcal{S}_1) \cup \mathrm{Loc}(\mathcal{S}_2)), \end{CD}$$

then the isomorphism $\pi_1(X(\mathcal{S})) \to N(\mathcal{S}) \times \pi_1(\mathrm{Loc}(\mathcal{S}))$ follows by the Seifert Van-Kampen Theorem. We now look at the homomorphism $\pi_1(U_{\mathcal{S}}) \to \pi_1(X(\mathcal{S}))$. We have a commutative diagram

$$\begin{array}{ccccc} N(\sigma(\mathcal{S}')) \times \pi_1(V_{\mathcal{S}'}) & \longrightarrow & N(\sigma(\mathcal{S}_1)) \times \pi_1(V_{\mathcal{S}_1}) & & \\ \downarrow & & \downarrow & \searrow^{p_1} & \\ N(\sigma(\mathcal{S}_2)) \times \pi_1(V_{\mathcal{S}_2}) & \longrightarrow & N(\sigma(\mathcal{S})) \times \pi_1(V_{\mathcal{S}}) & \dashrightarrow & \\ & \searrow_{p_2} & & & \pi_1(X(\mathcal{S})) \end{array}$$

where all the arrows are induced by inclusions and p_i is the composition

$$N(\sigma(\mathcal{S}_i)) \times \pi_1(V_{\mathcal{S}}) \to N(S_i) \times \pi_1(V_{\mathcal{S}_i}) \to N(S) \times \pi_1(\mathrm{Loc}(\mathcal{S}))$$

for each i. Then by the universal property of the push-out diagram we have a commutative diagram

$$\begin{CD} \pi_1(U_{\mathcal{S}}) @>{p}>> \pi_1(X(\mathcal{S})) \\ @V{\simeq}VV @V{\simeq}VV \\ N(\sigma(\mathcal{S})) \times \pi_1(V_{\mathcal{S}}) @>{p'}>> N(\mathcal{S}) \times \pi_1(\mathrm{Loc}(\mathcal{S})) \end{CD}$$

where p' is induced by the inclusions $N_{\sigma(\mathcal{S})} \subseteq N_{\mathcal{S}}$ and $V_{\mathcal{S}} \subseteq \mathrm{Loc}(\mathcal{S})$. The statement follows. □

Proof of Theorem 3. Reasoning as in the proof of Proposition 3.1.2, by gluing affine $\mathbb{T}$-varieties, it is enough to prove the affine case.

Let $\mathcal{D}$ be a p-divisor on $(Y, N_{\mathbb{Q}})$ such that $X(\mathcal{D})$ have log-terminal singularities. Let $r_{\mathcal{S}} \colon X(\mathcal{S}) \to \widetilde{X}(\mathcal{D})$ be a toroidal resolution of singularities of $\widetilde{X}(\mathcal{D})$ such that $\mathcal{S}$ is contraction-free and $\mathcal{D}$ has the same locus and support than $\mathcal{S}$, see [35]. Thus, we have a commutative diagram

$$\begin{array}{ccccc} & & X(\sigma(\mathcal{D})) \times V_{\mathcal{D}} & & \\ & \swarrow^{s_{\mathcal{D}}} & \downarrow i_{\mathcal{D}} & \searrow^{i_{\mathcal{S}}} & \\ X(\mathcal{D}) & \xleftarrow{r} & \widetilde{X}(\mathcal{D}) & \xleftarrow{r_{\mathcal{S}}} & X(\mathcal{S}) \end{array}$$

where $i_{\mathcal{D}}$ and $i_{\mathcal{S}}$ are inclusions, r and $r_{\mathcal{S}}$ are birational contractions and $s_{\mathcal{D}}$ is defined as the composition $r \circ i_{\mathcal{D}}$. Passing to the fundamental groups, $(r \circ r_{\mathcal{S}})_*$ is an isomorphisms by [**43**, Theorem1.1]. Being $X(\mathcal{S})$ smooth and $N(\mathcal{D}) = N(\mathcal{S})$ we have the isomorphisms $\pi_1(X(D)) \simeq \pi_1(X(\mathcal{S})) \simeq N(\mathcal{S}) \times \pi_1(\mathrm{Loc}(\mathcal{S})) = N(\mathcal{D}) \times \pi_1(\mathrm{Loc}(\mathcal{D}))$. Being $i_{\mathcal{D}}$ and $i_{\mathcal{S}}$ inclusions of open subsets we have that $(i_{\mathcal{D}})_*$ and $(i_{\mathcal{S}})_*$ are surjective. We deduce that the kernel of the three homomorphims $(s_{\mathcal{D}})_*$, $(i_{\mathcal{D}})_*$ and $(i_{\mathcal{S}})_*$ are equal. Thus $(s_{\mathcal{D}})_* \colon N(\sigma(\mathcal{D})) \times \pi_1(V_{\mathcal{D}}) \to N(\mathcal{D}) \times \pi_1(\mathrm{Loc}(\mathcal{D}))$ is $\alpha \times \beta$, where α is the surjection induced by the inclusion $N_{\sigma(\mathcal{D})} \subseteq N_{\mathcal{D}}$ and β is the surjection induced by the inclusion $V_{\mathcal{D}} \subseteq \mathrm{Loc}(\mathcal{D})$. □

3.2. The cohomology ring of a complexity one $\mathbb{T}$-variety

From now, we will consider divisorial fans $\mathcal{S}$ on $(\mathbb{P}^1, N_{\mathbb{Q}})$, such that $\widetilde{X}(\mathcal{S})$ is complete and $\mathbb{Q}$-factorial and describe the Cohomology ring and the Chow Ring under certain conditions (see Definition 3.2.9). Recall that using [**35**, Example 2.5] we can see in $\mathcal{S}$ when $\widetilde{X}(\mathcal{S})$ is a $\mathbb{Q}$-factorial variety. The results of this sections are generalizations of well-known Theorems, see [**23**, Chapter 5.].

Definition 3.2.1. Given a σ-polyhedron $\triangle \subseteq N_{\mathbb{Q}}$, we denote by $\mathcal{V}(\triangle)$ its set of vertices and by $\mathcal{N}(\triangle)$, or simply $\mathcal{N}$, its *normal fan* consisting of the regions where the function $\sigma^{\vee} \to \mathbb{Q}$, $u \mapsto \min_{v \in \triangle}\langle u, v\rangle$, is lineal. The cones of $\mathcal{N}$ are in one-to-one dimension-reversing correspondence with the faces $F \leq \triangle$ via the bijection

$$F \mapsto \lambda(F) := \{u \in M_{\mathbb{Q}} \mid \langle F, u\rangle = \min\langle \triangle, u\rangle\}.$$

Given a σ-polyhedron $\triangle$ we define the *affine toric bouquet* $X(\triangle) := \mathrm{Spec}(\mathbb{C}[\mathcal{N}])$, where $\mathbb{C}[\mathcal{N}] := \bigoplus_{u \in \sigma^{\vee} \cap M} \mathbb{C}\chi^u$ as a $\mathbb{C}$-vector space and the multiplication is given by

$$\chi^u \cdot \chi^{u'} := \begin{cases} \chi^{u+u'} & \text{if u,u' belong to a common cone of } \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The ring $\mathbb{C}[\mathcal{N}]$ is not an integral domain, hence $X(\triangle)$ is not a variety, but is a scheme, nevertheless since $\mathbb{C}[\mathcal{N}]$ is reduced, $X(\triangle)$ have a decomposition into a finite union of irreducible toric varieties

$$X(\triangle) = \bigcup_{v \in \mathcal{V}(\triangle)} X(\mathbb{Q}_{\geq 0} \cdot (\triangle - v)).$$

These irreducible toric varieties are intersecting along $\mathbb{T}$-invariant divisors, hence the big torus is still acting on $X(\triangle)$. Observe that the orbits of $X(\triangle)$ are in one-to-one dimension-reversing correspondence with the faces $F \leq \triangle$. We denote by $\triangle(k)$ the set of faces of codimension k and given a face $F \in \triangle(k)$ we denote by $\mathcal{O}_F$ its corresponding torus orbit of dimension k. Given Δ a polyhedral complex consisting of a finite number of σ_i-polyhedra $\triangle_i$, we can glue the affine toric bouquets $X(\triangle_i)$ and $X(\triangle_j)$ along $X(\triangle_i \cap \triangle_j)$ to obtain a *toric bouquet* $X(\Delta)$. We denote by $\Delta(k)$ the set of faces of codimension k of Δ.

Proposition 3.2.2. *The Chow group $A_k(X)$ of an arbitrary toric bouquet $X = X(\Delta)$ is generated by the classes of the orbit closures $[\overline{\mathcal{O}_F}]$ for $F \in \Delta(k)$.*

Proof. We denote bye X_i the union of all orbits corresponding to faces of codimension at most i. This gives a filtration $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_{-1} = \emptyset$ by closed subschemes. The complement of X_{i-1} in X_i is a disjoint union of $|\Delta(i)|$ torus orbits of dimension i. We conclude by the exact sequence [**23**, Proposition 1.8]

$$A_k(X_{i-1}) \to A_k(X_i) \to A_k(X_i - X_{i-1}) \to 0,$$

and by induction on i. □

Definition 3.2.3. In $\widetilde{X}(\mathcal{S})$ we have two kind of $\mathbb{T}$-invariant cycles. Given a point $y \in Y$ and a face F of $\mathcal{S}_y$ we have a $\mathbb{T}$-invariant cycle $[\overline{\mathcal{O}_F}] \in A_*(\widetilde{X}(\mathcal{S}))$. We call such cycle *vertical* if y is a closed point of Y and *horizontal* if y is the generic point of Y. If $y \in Y$ is the generic point and $\rho \in \mathcal{S}_y = \Sigma(\mathcal{S})$ is a ray, we denote the corresponding *horizontal divisor* by D_ρ and if $y \in Y$ is a closed point and $v \in \mathcal{V}(\mathcal{S}_y)$ is a vertex, we denote the corresponding *vertical divisor* by $D_{(v,y)}$.

Proposition 3.2.4. *Given a divisorial fan $\mathcal{S}$ on a curve Y the Chow group $A_k(\widetilde{X}(\mathcal{S}))$ is generated by the classes of the horizontal and vertical cycles of dimension k.*

Proof. Let $p_1, \ldots, p_r$ be the support of the divisorial fan $\mathcal{S}$. Recall that each $\pi^{-1}(p_i)$ is a toric bouquet and that we have an isomorphism

$$\pi^{-1}(Y - \{p_1, \ldots, p_r\}) \simeq X(\Sigma(\mathcal{S})) \times (Y - \{p_1, \ldots, p_r\}).$$

Here $X(\Sigma(\mathcal{S}))$ denotes the toric variety associated to the fan $\Sigma(\mathcal{S})$. Recall that by Proposition 3.2.2 the Chow group $A_k(\bigcup_{i=1}^r \pi^{-1}(p_i))$ is generated by vertical cycles of dimension k, while using the above isomorphism we see that the Chow group $A_k(\pi^{-1}(Y - \{p_1, \ldots, p_r\}))$ is generated by vertical and horizontal cycles of dimension k. Using the exact sequence of Chow groups relating the closed subscheme $\bigcup_{i=1}^r \pi^{-1}(p_i)$ and its complement the result follows. □

Definition 3.2.5. Given a polyhedral complex Δ on $N_\mathbb{Q}$, we say that Δ is *shellable* if we can order the maximal polyhedra $\triangle_1, \ldots, \triangle_k$ of Δ such that for each i the following set odered by inclusion

$$\{F \leq \triangle_i \mid F \text{ is not contained in } \cup_{j<i} \triangle_j\},$$

has a unique minimal element denoted by G_i.

For example the fan of a projective and simplicial toric variety is always shellable by [**23**, Section 5.2, Lemma]. Observe that for each face F of a shellable polyhedral complex Δ there is a unique i such that $G_i \subseteq F \subseteq \triangle_i$.

Definition 3.2.6. We say that a polyhedral complex Δ is *simplicial* if for each vertex $v \in \Delta$ and maximal polyhedron $v \in \triangle \in \Delta$ the cone $\mathbb{Q} \cdot (v - \Delta)$ is simplicial.

Let Δ be a shellable polyhedral complex and $\triangle_1, \ldots, \triangle_k$, the order induced in its maximal polyhedra. for $1 \leq i \leq k$ we define the subvarieties of $X(\Delta)$

$$Y_i := \bigcup_{G_i \subseteq F \subseteq \triangle_i} \mathcal{O}_F, \qquad Z_i := Y_i \cup Y_{i+1} \cup \cdots \cup Y_k.$$

This is a generalization of the shellability condition on fans given in [**23**, Section 5.2] which is related to the shellability problem for cones [**10**].

Lemma 3.2.7. *Each Z_i is closed, $Z_1 = X(\Delta)$ and $Z_i - Z_{i+1} = Y_i$. Moreover, if Δ is simplicial then each Y_i is the quotient of an affine space by a finite group.*

Proof. Recall that for each face F of $\mathcal{S}_y$ there is a unique i such that $G_i \subseteq F \subseteq \triangle_i$, then $\pi^{-1}(y)$ is the disjoint union of the sets Y_i. The closure of $\mathcal{O}_F$ is the union of all the orbits $\mathcal{O}_{F'}$ with $F' \supseteq F$, then we conclude that each Z_i is closed. The last assertion follows from the fact that each irreducible toric component of $\pi^{-1}(y)$ is a simplicial toric variety. $\square$

Proposition 3.2.8. *Let Δ be a shellable and simplicial polyhedral complex with complete support. Then the classes $[\overline{\mathcal{O}_{G_i}}]$ form a basis for $A_*(X(\Delta))_{\mathbb{Q}} \simeq H_{2*}(X(\Delta); \mathbb{Q})$ and $H_q(X(\Delta)) = 0$ for q odd.*

Proof. In the proof all homologies and Chow groups are over $\mathbb{Q}$. Recall that $X(\Delta) = Z_1$. We prove, by descending induction on i, that the canonical map $A_*(Z_i) \to H_*(Z_i)$ is an isomorphism, that the classes $[\overline{\mathcal{O}_{G_j}}]$, for any $j \geq i$, form a basis of $A_*(Z_i)$ and that $H_q(Z_i) = 0$ for q odd. Following Fulton [**23**, Pag. 103] we have a commutative diagram of Chow groups and Borel-Moore homology with rational coefficients

$$\begin{array}{ccccccccccc} & & A_p(Z_{i+1}) & \longrightarrow & A_p(Z_i) & \longrightarrow & A_p(Y_i) & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ \dots \to H_{2p+1}(Y_i) & \to & H_{2p}(Z_{i+1}) & \to & H_{2p}(Z_i) & \to & H_{2p}(Y_i) & \to & H_{2p-1}(Z_{i+1}) & \to & \dots \end{array}$$

Since Y_i is the quotient of an affine space by a finite group we conclude that $A_k(Y_i) \simeq H_{2k}(Y_i) \simeq \mathbb{Q}$ is generated by the class of Y_i for $k = \dim_{\mathbb{C}} Y_i$ and otherwise both spaces are trivial. By induction hypothesis $H_q(Z_{i+1}) = 0$ for q odd. The statement follows. $\square$

Consider a divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$, with support $\{p_1, \dots, p_r\}$, such that $\widetilde{X}(\mathcal{S})$ is complete and $\mathbb{Q}$-factorial. We denote by $\mathcal{S}_i$ the polyhedral complex corresponding to the toric bouquet $\pi^{-1}(p_i)$, for $i \in \{1, \dots, r\}$ and by $\mathcal{S}_0$ the polyhedral complex corresponding to the toric variety $\pi^{-1}(p_0)$, where p_0 is a general point. Assume that $\mathcal{S}_i$ is shellable with ordered maximal polyhedra $\triangle_1^i, \dots, \triangle_{k_i}^i$ and minimal elements $G_1^i, \dots, G_{k_i}^i$ for any i. We denote by

$$B_i := \{[\overline{\mathcal{O}_{G_j^i}}] : j \in \{1, \dots, k_i\}\}$$

the $\mathbb{Q}$-vector space basis of $A_k(X(\mathcal{S}_i))$. For each face F of $\mathcal{S}_0$ we denote by $\mathcal{S}(F)_i$ the set of faces of $\mathcal{S}_i$ which have the same tailcone and dimension of F. For each $F' \in \mathcal{S}(F)_i$ we denote by $v(F')$ its unique vertex. We define the 0-graded linear map of $\mathbb{Q}$-vector spaces

$$j_i \colon A_*(X(\mathcal{S}_0)) \to A_*(X(\mathcal{S}_i)), \qquad [\overline{\mathcal{O}_{G_j^0}}] \mapsto \sum_{F' \in \mathcal{S}(G_j^0)_i} \mu(v(F'))[\overline{\mathcal{O}_F}],$$

for $j \in \{1, \dots, i_0\}$. Observe that j_i is well defined, since B_0 is a $\mathbb{Q}$-vector space basis of $A_*(X(\mathcal{S}_0))$ by Proposition 3.2.8.

Definition 3.2.9. We say that $\mathcal{S}$ is *complete* if $\widetilde{X}(\mathcal{S})$ is a complete variety. We say that $\mathcal{S}$ is *simplicial* if $\widetilde{X}(\mathcal{S})$ is a $\mathbb{Q}$-factorial variety. We say that $\mathcal{S}$ is *shellable* if $j_{i,k}$ is injective for each $k \in \mathbb{N}$ and $i \in \{1, \dots, r\}$.

For each face F of $\mathcal{S}_i$, we denote by $i(F)$ the minimum positive integer such that $F \subseteq \triangle_{i(F)}$. Observe that a sufficient condition for $j_{i,k}$ to be injective is that for each $1 \leq n < m \leq k_0$ we have that

$$\min\{i(F) \mid F \in \mathcal{S}(G_n^0)_i\} < \min\{i(F) \mid F \in \mathcal{S}(G_m^0)_i\}.$$

Indeed in this case the representative matrix of j_i with respect to the basis B_0 of $A_*(X(\mathcal{S}_0))$ and B_i of $A_*(X(\mathcal{S}_i))$ contains a triangular submatrix of full rank.

Notation 3.2.10. Given a complete, simplicial and shellable divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$, we denote by $\mathbb{Q}[D_\rho, D_{(p,v)}]$ the polynomial ring over $\mathbb{Q}$ generated by elements D_ρ and $D_{(p,v)}$ for $\rho \in \Sigma(\mathcal{S})(1)$ and $v \in \mathcal{V}(\mathcal{S}_p)$, with $p \in \mathbb{P}^1$.

For each face F of a slice $\mathcal{S}_p$ with $p \in \mathbb{P}^1$, we have a corresponding monomial in $\mathbb{Q}[D_\rho, D_{(p,v)}]$, which is the product of all the elements D_ρ and $D_{(p,v)}$ such that $v \in F$ and $\rho \in \sigma(F)$. We denote such element by $p(F)$.

For each cone σ of the fan $\Sigma(\mathcal{S})$, we have a corresponding monomial in $\mathbb{Q}[D_\rho, D_{(p,v)}]$, which is the product of all the elements D_ρ with $\rho \in \sigma$. We denote such element by $p(\sigma)$.

We denote by I the ideal generated by

$$\text{div}(f\chi^u) := \sum_{\rho \in \Sigma(\mathcal{S})(1)} \langle \rho, u \rangle D_\rho + \sum_{p \in \mathbb{P}^1, v \in \mathcal{V}(\mathcal{S}_p)} \mu(v)(\langle v, u \rangle + ord_p f) \cdot D_{(p,v)}, \tag{3.2.1}$$

for each rational function $f \in \mathbb{C}(\mathbb{P}^1)$ and character lattice $\chi^u \in M$ and by all the monomials which are not of the form $p(F)$ nor $p(\sigma)$.

Proof of Theorem 4. In the proof we omit the rational coefficients in order to abbreviate notation. We start by proving the first statement. Let $U = \widetilde{X}(\mathcal{S}) - \cup_{i=1}^r \pi^{-1}(p_i)$. We have a commutative diagram of Chow groups and Borel-Moore homology with rational coefficients

$$\begin{array}{ccccccccc}
 & & A_k(U^c) & \longrightarrow & A_k(\widetilde{X}(\mathcal{S})) & \longrightarrow & A_k(U) & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \\
H_{2k+1}(U) & \longrightarrow & H_{2k}(U^c) & \longrightarrow & H_{2k}(\widetilde{X}(\mathcal{S})) & \longrightarrow & H_{2k}(U) & \longrightarrow & H_{2k-1}(U^c).
\end{array}$$

The first vertical arrow is an isomorphism by Proposition 3.2.8. Using that $A_k(\mathbb{P}^1 - \{p_1, \ldots, p_r\}) \to H_{2k}(\mathbb{P}^1 - \{p_1, \ldots, p_r\})$ is an isomorphism for each k, Künneth formula for Borel-Moore homology [**13**, Section 2.6.19] and Proposition 3.2.8, we see that the third vertical arrow is an isomorphism. Moreover $H_{2k-1}(U^c) = 0$ by Proposition 3.2.8. We define an injective homomorphism

$$A_*(X(\mathcal{S}_0)) \to A_*(U^c)$$

in the following way. Given the orbit closure V of $X(\mathcal{S}_0)$ we map its class $[V] \in A_*(X(\mathcal{S}_0))$ to the class of $\imath_i^*(\overline{V \times \mathbb{P}^1 - \{p_1, \ldots, p_r\}})$, where $\imath_i \colon X(\mathcal{S}_i) \to \widetilde{X}(\mathcal{S})$ is the inclusion of the reduced fiber. This map extends to an injective homomorphism

$$i \colon \mathbb{Q}^{r-1} \otimes_{\mathbb{Q}} A_*(X(\mathcal{S}_0)) \to A_*(U^c) \simeq \bigoplus_{i=1}^r A_*(X(\mathcal{S}_i))$$

defined by

$$(m_1, \ldots, m_{r-1}) \otimes_{\mathbb{Q}} [V] \mapsto (m_1 j_1([V]), \ldots, m_{r-1} j_{r-1}([V]), -m j_r([V])),$$

where $m = m_1 + \cdots + m_{r-1}$. The image of i is in the kernel of $A_*(U^c) \to A_*(\widetilde{X}(\mathcal{S}))$. Indeed, $(m_1, \ldots, m_{r-1}) \otimes_{\mathbb{Q}} [V]$ is mapped to the divisor of a rational function $f\chi^0$, where f has order m_i at p_i, for $i \in \{1, \ldots, r-1\}$ and order $-\sum_{i=1}^{r-1} m_i$ at p_r. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathbb{Q}^{r-1} \otimes_{\mathbb{Q}} A_k(X(\mathcal{S}_0)) & & & & & & \\
 & & \downarrow i & & & & & & \\
0 & \longrightarrow & K & \longrightarrow & A_k(U^c) & \longrightarrow & A_k(\widetilde{X}(\mathcal{S})) & \longrightarrow & A_k(U) \longrightarrow 0 \\
 & & \downarrow j & & \downarrow \simeq & & \downarrow & & \downarrow \simeq \\
 & & \mathbb{Q}^{r-1} \otimes_{\mathbb{Q}} H_k(X(\mathcal{S}_0)) & \longrightarrow & H_{2k}(U^c) & \longrightarrow & H_{2k}(\widetilde{X}(\mathcal{S})) & \longrightarrow & H_{2k}(U) \longrightarrow 0,
\end{array}$$

where K is the kernel of $A_k(U^c) \to A_k(\widetilde{X}(\mathcal{S}))$, i is the injection induced by the above homomorphism, and j is the unique arrow making the diagram commutative. By the four lemma the third vertical arrow is surjective, then j is injective. Since

$$\dim(A_k(X(\mathcal{S}_0)))_{\mathbb{Q}} = \dim(H_k(X(\mathcal{S}_0)); \mathbb{Q})$$

for each k by Proposition 3.2.8, all the vertical arrows are isomorphisms. □

Now we prove three Lemmas concerning the structure of the ring $\mathbb{Q}[D_\rho, D_{(p,v)}]$, in order to prove Theorem 5. Recall that given a complete, simplicial and shellable divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$ with support $\{p_1, \ldots, p_r\}$ we denote by $\triangle_1^i, \ldots, \triangle_{k_i}^i$ the ordered maximal polyhedra of $\mathcal{S}_i$ and we denote by $G_1^i, \ldots, G_{k_i}^i$ the minimal elements for any i. Moreover, we will denote by $\triangle_1, \ldots, \triangle_k$, the ordered maximal cones of $\Sigma(\mathcal{S})$ and we denote by $G_1, \ldots, G_k$ the minimal elements. We assume that the shellability induced in $\mathcal{S}_0$ and $\Sigma(\mathcal{S})$ are the same.

Lemma 3.2.11. *Let $\mathcal{S}$ be a divisorial fan on $\mathbb{P}^1$ as above. Let $\emptyset \neq G \subsetneq H \subseteq F$ be faces in $\mathcal{S}_i$, with $i \in \{0, \ldots, r\}$. Then there exists an element on I of the form*

$$p(H) - \sum_j m_j p(H_j),$$

where H_j are faces of $\mathcal{S}_i$, with with $G \subseteq H_j$ and $H_j \not\subseteq F$, and $m_j \in \mathbb{Q}$.

Proof. Let $\mathcal{V}(F) = \{v_1 \ldots, v_k\}$, and $\sigma(F) = \{\rho_1, \ldots, \rho_{k'}\}$. We prove the case where $v_1 \in \mathcal{V}(H)$ is not in $\mathcal{V}(G)$ (the case where exists $\rho \in \sigma(H)$, not in $\sigma(G)$ is analogous). Recall that the set

$$S := \{(\rho_1, 0), \ldots, (\rho_{k'}, 0), (v_1, 1), \ldots, (v_k, 1)\},$$

is linearly independent in $N_{\mathbb{Q}} \oplus \mathbb{Q}$. Consider an element $u' \in M \oplus \mathbb{Z} = \mathrm{Hom}(N \oplus \mathbb{Z}, \mathbb{Z})$ which vanishes at all the elements of $S - \{(v_1, 1)\}$ and $u'(v_1, 1) = 1$. Let u be the restriction of u' to M and consider a rational function $f \in \mathbb{C}(\mathbb{P}^1)$ with $\mathrm{ord}_{p_i}(f) = u'(0,1)$ and $\mathrm{ord}_{p_k}(f) = 0$ for $k \neq i$. Recall that $\mathrm{div}(f\chi^u)$ is in I by definition. Moreover, by (3.2.1) we have

$$\begin{aligned}
\mathrm{div}(f\chi^u) := & \sum_{\rho \in \sigma(F)^c} \langle \rho, u \rangle D_\rho + \sum_{v \in \mathcal{V}(F)^c} \mu(v)(\langle v, u \rangle + ord_{p_i} f) \cdot D_{(p_i, v)} \\
& + D_{(p_i, v_1)} + \sum_{p \in \mathbb{P}^1 - \{p_i\}, v \in \mathcal{V}(\mathcal{S}_p)} \mu(v)(\langle v, u \rangle + ord_p f) \cdot D_{(p,v)},
\end{aligned}$$

where all but the summand $D_{(p_i,v_1)}$ correspond to vertices or rays which are not in $\mathcal{V}(F)\cup\sigma(F)$. Multiplying $\operatorname{div}(f\chi^u)$ by $p(H)/D_{(p_i,v_1)}$, subtracting all the monomials which are not of the form $p(F)$, with F a face of $\mathcal{S}_i$, and observing that $p(G)$ divides $p(H)/D_{(p_i,v_1)}$, we get the result. $\square$

LEMMA 3.2.12. *Let $\mathcal{S}$ be a divisorial fan as above and let $\mathcal{S}_i$ be a slice with $i \in \{0,\dots,r\}$. The ideal $I_i := \langle D_{(p,v)} + I \mid v \in \mathcal{S}_i\rangle$ of $\mathbb{Q}[D_\rho, D_{(p,v)}]/I$ is generated as a $\mathbb{Q}$-vectorial space by the set $\{p(G^i_j) + I \mid j \in \{1,\dots,k_i\}\}$.*

PROOF. First we prove that I_i is generated by square-free monomials. To this aim it is enough to show that for any face F of $\mathcal{S}_i$ and $v \in \mathcal{V}(F)$ the element $D_{(p_i,v)}p(F)$ and $D_\rho p(F)$ are equivalent to sum of square-free monomials modulo I. Following the proof of Lemma 3.2.11 we can find an element $\operatorname{div}(f\chi^u) \in I$ which contains $D_{(p_i,v)}$ as a summand and the other summands corresponds to vertices and rays which are not in F. Multiplying this element by $p(F) + I$ we conclude that $D_{(p_i,v)}p(F)$ is equal to a linear combination of square-free monomials. A similar argument applies to $D_\rho p(F)$.

By descending induction on k, we prove that if $G^i_k \subseteq H \subseteq F_k$ then $p(H) + I$ is in the submodule generated by $p(G^i_j) + I$ with $j \geq k$. If $H = G^i_k$ we are done, otherwise we can apply Lemma 3.2.11 to $\emptyset \neq G^i_k \subsetneq H \subseteq F_k$ and conclude by the induction hypothesis. $\square$

In the following proof, given a face $F \in \mathcal{S}_{p_i}$, with $i \in \{1,\dots,r\}$, we will denote by $\mathcal{D}_F$ the p-divisor with support $\{p_1,\dots,p_r\}$ and slices $\mathcal{D}_{F\,p_i} = F$ and $\mathcal{D}_{F\,p_j} = \emptyset$, if $j \neq i$. Recall that we have a open embedding $X(\mathcal{D}_F) \to \widetilde{X}(\mathcal{S})$.

LEMMA 3.2.13. *Let $\mathcal{S}$ be a complete, simplicial and shellable divisorial fan on $\mathbb{P}^1$. Then every $\mathbb{T}$-invariant prime subvariety of $\widetilde{X}(\mathcal{S})$ is the complete intersection of $\mathbb{T}$-invariant divisors.*

PROOF. Recall that, according to Definition 3.2.3, any $\mathbb{T}$-invariant prime subvariety of $\widetilde{X}(\mathcal{S})$ is of the form $\overline{\mathcal{O}_F}$, where F is a face of the slice $\mathcal{S}_y$ and $y \in \mathbb{P}^1$ is a point. If y is a closed point, then $\overline{\mathcal{O}_F}$, with $F \in \mathcal{S}_y$, equals $\bigcup_{F'\supset F}\mathcal{O}_{F'}$. Observe that $F' \supset F$ if and only if $\sigma(F') \supset \sigma(F)$ and $\mathcal{V}(F') \supset \mathcal{V}(F)$, so that the following equality holds:

$$\overline{\mathcal{O}_F} = \bigcap_{v\in\mathcal{V}(F)} D_{(y,v)} \cap \bigcap_{\rho\in\sigma(F)} D_\rho.$$

If y is the generic point and $p \in \mathbb{P}^1$ is a closed point, localizing to a toric neighbourhood of p we have the equality

$$\overline{\mathcal{O}_F} \cap \pi^{-1}(p) = \bigcup_{F'\in\mathcal{S}_p,\sigma(F')\supset\sigma(F)} \mathcal{O}_{F'},$$

where the $\subseteq$ inclusion is clear, while the $\supseteq$ inclusion is due to the fact that if $F' \in \mathcal{S}_p$ and $\sigma(F') \not\supset \sigma(F)$, then $X(\mathcal{D}_{F'})$ is an open subset of $\widetilde{X}(\mathcal{S})$ which contains $\mathcal{O}_{F'}$ and is disjoint from $\overline{\mathcal{O}_F}$. We conclude that

$$\overline{\mathcal{O}_F} = \bigcup_{p\in\mathbb{P}^1}\ \bigcup_{F'\in\mathcal{S}_p,\sigma(F')\supset\sigma(F)} \mathcal{O}_{F'} = \bigcap_{\rho\in\sigma(F)} \overline{\mathcal{O}_\rho} = \bigcap_{\rho\in\sigma(F)} D_\rho.$$

$\square$

Proof of Theorem 5. We have a canonical homomorphism

$$\mathbb{Q}[D_\rho, D_{(p,v)}] \to A^*(\widetilde{X}(\mathcal{S})), \qquad D \mapsto [D]$$

which maps every D_ρ (resp. $D_{(p,v)}$) to the class of the corresponding divisor. In $\widetilde{X}(\mathcal{S})$ every $\mathbb{T}$-invariant cycle is the intersection of $\mathbb{T}$-invariant divisors by Lemma 3.2.13, then the above homomorphism is surjective. Recall by [**5**, Theorem 26] that given a rational function f of $\mathbb{P}^1$ and a character χ^u of the torus acting on $\widetilde{X}(\mathcal{S})$, the element $\operatorname{div}(f\chi^u) \in \mathbb{Q}[D_\rho, D_{(p,v)}]$ is in the kernel of the homomorphism. Consider a monomial m of the form

$$D_{\rho_1} \dots D_{\rho_{k'}} D_{(p_1,v_1)} \dots D_{(p_k,v_k)} \in \mathbb{Q}[D_\rho, D_{(p,v)}].$$

If there exist $1 \leq i < j \leq k$ such that $p_i \neq p_j$ then the image of m in $A^*(\widetilde{X}(\mathcal{S}))$ is zero because we are intersecting two divisors which are in diferent fibers of the quotient morphism $\widetilde{X}(\mathcal{S}) \to \mathbb{P}^1$. If all the p_i's are equal, we can localize to the toric neighbourhood $X(\Sigma_i)$ of $\pi^{-1}(p_i)$ and see that there exists no face F in $\mathcal{S}_{p_i}$ such that $\sigma(F) = \langle \rho_1, \dots, \rho_{k'} \rangle$ and $\mathcal{V}(F) = \{v_1, \dots, v_k\}$ if and only if there exists no cone in Σ_i generated by $\{(\rho_1, 0), \dots, (\rho_{k'}, 0), (v_1, 1), \dots, (v_k, 1)\}$. The last condition is equivalent to ask for the divisors in m to have empty intersection. If this is the case then m is in the kernel of the homomorphism. If there is no $D_{(p_i,v_i)}$ in the monomial, and there is no cone $\sigma \in \Sigma(\mathcal{S})$ such that $\rho_1, \dots, \rho_{k'}$ generates σ, then the horizontal divisors in m have empty intersection by [**23**, Section 5.2]. Thus we have a well-defined surjective homomorphism

$$\phi \colon \mathbb{Q}[D_\rho, D_{(p,v)}]/I \to A^*(\widetilde{X}(\mathcal{S})).$$

In order to conclude we prove that there is a subset of $\mathbb{Q}[D_\rho, D_{(p,v)}]/I$ which generates it as a $\mathbb{Q}$-vector space and whose image is a $\mathbb{Q}$-basis of the Chow ring.

Observe that the subring of $A^*(\widetilde{X}(\mathcal{S}))_{\mathbb{Q}}$ generated by $\{D_\rho \mid \rho \in \Sigma(\mathcal{S})(1)\}$ is isomorphic to $A^*(X(\Sigma(\mathcal{S})))_{\mathbb{Q}}$, and by Proposition [**23**, Chapter 5] it is generated as $\mathbb{Q}$-vectorial space by the monomials $p(G_j)$, with $j \in \{1, \dots, k\}$.

From now, we use the notation of the proof of Theorem 5. The restriction of ϕ to I_i induces a surjective homomorphism $\alpha_i \colon I_i \to A^*(X(\mathcal{S}_i))$ of $\mathbb{Q}$-vector spaces which maps $\{p(G^i_j) + I \mid j \in \{1, \dots, k_i\}\}$ into a $\mathbb{Q}$-basis of the codomain. Then α_i is an isomorphism by 3.2.12. Consider a rational function $f \in \mathbb{C}(\mathbb{P}^1)$ with $\operatorname{ord}_{p_0}(f) = 1$ and $\operatorname{ord}_{p_i}(f) = -1$. Expanding the product $p(G_j)\operatorname{div}(f\chi^0) \in I$ we deduce the following

$$p(G^0_j) + I = \sum_{v \in \mathcal{V}(\mathcal{S}_i)} \mu(v) p(G_j) D_{(p_i,v)} + I.$$

For each i, we define the $\mathbb{Q}$-linear map

$$I_0 \to I_i. \qquad p(F) + I \mapsto \sum_{v \in \mathcal{V}(\mathcal{S}_i)} \mu(v) p(F) D_{(p_i,v)} + I.$$

which makes commute the following diagram

$$\begin{array}{ccccc}
I_0 & \hookrightarrow & I_i & \hookrightarrow & \mathbb{Q}[D_\rho, D_{(p,v)}]/I \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \phi \\
A_*(X(\mathcal{S}_0)) & \xhookrightarrow{j_i} & A_*(X(\mathcal{S}_i)) & \hookrightarrow & A_*(\widetilde{X}(\mathcal{S})).
\end{array}$$

In particular, for each i we can choose a set of elements $\mathcal{B}_i$ of I_i which are linear combination of the monomials $p(G^i_j)+I$, such that the image of $\mathcal{B}_i$ via ϕ is a basis of $\operatorname{coker}(j_i)$. Observe that $\{p(G^0_j)+I \mid j\in\{1,\ldots,k_0\}\}\cup\mathcal{B}_i$ generates I_i as a $\mathbb{Q}$-vector space for each i. Being $\widetilde{X}(\mathcal{S})$ a $\mathbb{Q}$-factorial variety and using Theorem 5 we have that

$$A^*(\widetilde{X}(\mathcal{S}))\simeq A_*(\widetilde{X}(\mathcal{S}))\simeq A_{*-2}(X(\Sigma(\mathcal{S})))\oplus A_*(X(\mathcal{S}_0))\bigoplus_{i=1}^{r}\operatorname{coker}(j_i).$$

We conclude that the set

$$\{p(G_j)+I \mid j\in\{1,\ldots k\}\}\cup\{p(G^0_j)+I \mid j\in\{1,\ldots,k_0\}\}\bigcup_{i=1}^{r}\mathcal{B}_i,$$

is a subset of $\mathbb{Q}[D_\rho,D_{(p,v)}]/I$ which generates its as a $\mathbb{Q}$-vector space and whose image is a $\mathbb{Q}$-basis of the Chow ring. □

The following example gives the dimensions of cohomology groups with rational coefficients of a complete $\mathbb{Q}$-factorial threefold with shellable divisorial fan.

Example 3.2.14. Let $\mathcal{S}$ be shellable divisorial fan on $\mathbb{P}^1$ such that $\widetilde{X}(\mathcal{S})$ is a complete $\mathbb{Q}$-factorial threefold. Denote by $p_1,\ldots,p_r$ the support of $\mathcal{S}$ and by $\mathcal{S}_i$ the slice at the point p_i. We denote by s_i the number of maximal polyhedra in $\mathcal{S}_i$. Using Theorem 5 we can compute the dimension of its cohomology groups with rational coefficients

$$\dim_{\mathbb{Q}}(H^0(\widetilde{X}(\mathcal{S})))=1,\qquad \dim_{\mathbb{Q}}(H^2(\widetilde{X}(\mathcal{S})))=\sum_{i=1}^{r}|\mathcal{V}(\mathcal{S}_i)|+|\Sigma(\mathcal{S})|-r-2,$$

$$\dim_{\mathbb{Q}}(H^4(\widetilde{X}(\mathcal{S})))=\sum_{i=1}^{r}(s_i-|\mathcal{V}(\mathcal{S}_i)|)-r|\Sigma(\mathcal{S})|+r+1,\qquad \dim_{\mathbb{Q}}(H^6(\widetilde{X}(\mathcal{S})))=1.$$

Remark 3.2.15. Observe that the conclusion of Lemma 3.2.13 is no longer true if we substitute $\widetilde{X}(\mathcal{S})$ with $X(\mathcal{S})$. As an example consider the quadric $Q=V(x_1x_2+x_3x_4+x_5x_6)$ of $\mathbb{P}^5$. It admits an effective action of $(\mathbb{K}^*)^3$ and thus is a $\mathbb{T}$-variety of complexity one, so that $Q=X(\mathcal{S})$ for some divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$. On the other hand Q is isomorphic to the Plücker embedding of the Grassmannian $G(2,4)$ and its Chow ring of Q is well known: $A^1(X)\cong\mathbb{Z}$ is generated by the class of a hyperplane section and $A^2(X)\cong\mathbb{Z}^2$. Hence $A^*(X)$ cannot be generated by classes of invariant divisors.

3.3. Singular Cohomology of complexity-one $\mathbb{T}$-varieties

In this section Y denotes a smooth curve. Let $\mathcal{D}=\sum_{i=1}^{r}\Delta_{p_i}\otimes p_i$ denote a p-divisor on a curve Y with tailcone σ. We define the *degree* of $\mathcal{D}$ as

$$\deg(\mathcal{D}):=\sum_{i=1}^{r}\Delta_{p_i}\subseteq N_{\mathbb{Q}}.$$

For a p-divisor $\mathcal{D}$ on a curve Y we have that $\deg(\mathcal{D})\neq\emptyset\Leftrightarrow\operatorname{Loc}(\mathcal{D})=Y$. If $\mathcal{D}$ is a polyhedral divisor on Y then $\mathcal{D}$ is a p-divisor if and only if $\deg(\mathcal{D})\subsetneq\sigma$ and $\mathcal{D}(u)$ has a principal multiple for all $u\in\sigma^\vee$ with $u^\perp\cap(\deg(\mathcal{D}))\neq 0$.

Now we turn to compute the higher homology and cohomology of contraction-free complexity-one $\mathbb{T}$-varieties. Given $\mathcal{D}=\sum_{i=1}^{r}\Delta^r_{p=1}\oplus p_i$ be a p-divisor on a curve Y. we consider the open set $V := \coprod_{i=1}^r V_i := \coprod_{i=1}^r \mathbb{C}$ which is an analytic neighbourhood of the support of $\mathcal{D}$. We will write $U := \pi^{-1}(V)$ and $U_i := \pi^{-1}(V_i)$ for each i. We recall that we have an equality $U_i := X(\sigma_i)$ for some cone σ_i in $N'_{\mathbb{Q}} := N_{\mathbb{Q}} \oplus \mathbb{Q}$. For each polyhedra Δ_i we denote by $N(\Delta_i) := N/N_{\Delta_i}$, where $N_{\Delta_i} := \mathbb{Z} \cap \{v_1 - v_2 \mid v_1, v_2 \in \Delta_i\}$. For each i we have that $N'(\sigma_i) \simeq N(\Delta_i)$ and we will denote by N_{Δ_i} the kernel of the surjective homomorphism $N(\sigma) \to N(\Delta_i)$, then we can identify $N(\Delta_i)$ with a subspace of $N(\sigma)$ via the isomorphism $N(\sigma) \simeq N(\Delta_i) \oplus N_{\Delta_i}$ for each i. We will use the following notation

$$\begin{aligned} N_i^{\mathcal{D}}(1) &:= N_{\Delta_i} \cap \langle N_{\Delta_{i+1}}, \dots, N_{\Delta_r} \rangle, \\ N_i^{\mathcal{D}}(2) &:= N_{\Delta_i} \cap N(\Delta_{i+1}) \cap \cdots \cap N(\Delta_r), \\ N_i^{\mathcal{D}}(3) &:= N(\Delta_i) \cap \langle N_{\Delta_{i+1}}, \dots, N_{\Delta_r} \rangle, \\ N_i^{\mathcal{D}}(4) &:= N(\Delta_i) \cap N(\Delta_{i+1}) \cap \cdots \cap N(\Delta_r). \end{aligned}$$

Observe that for each i the subspaces $N_i^{\mathcal{D}}(1), N_i^{\mathcal{D}}(2), N_i^{\mathcal{D}}(3)$ and $N_i^{\mathcal{D}}(4)$ are pairwise disjoint and generate $N(\sigma)$. To abreviate notation, given natural numbers a_1, a_2, a_3, a_4 we will write

$$\wedge^{a_1,a_2,a_3,a_4} N_i^{\mathcal{D}} := \wedge^{a_1} N_i^{\mathcal{D}}(1) \otimes_{\mathbb{Z}} \wedge^{a_2} N_i^{\mathcal{D}}(2) \otimes_{\mathbb{Z}} \wedge^{a_3} N_i^{\mathcal{D}}(3) \otimes_{\mathbb{Z}} \wedge^{a_4} N_i^{\mathcal{D}}(4),$$

and

$$H_k(N^{\mathcal{D}}) := \bigoplus_{i=1}^{r-1} \left(\bigoplus_{(a_1,a_2,a_3,a_4) \in \mathcal{I}_{k-1}} \wedge^{a_1,a_2,a_3,a_4} N_i^{\mathcal{D}} \right),$$

where $\mathcal{I}_k := \{(a_1, a_2, a_3, a_4) \in \mathbb{N}^4 \mid a_1 + a_2 + a_3 + a_4 = k, (a_1 + a_2)(a_1 + a_3) \neq 0\}$.

Proposition 3.3.1. *Let $\mathcal{D} = \Delta_p \oplus p$ be a p-divisor on a curve Y. Then the following statements holds.*

(1) *If $\mathcal{D}$ have affine locus, then $H_k(\widetilde{X}(\mathcal{D}))$ is isomorphic to*

$$\wedge^k N_1^{\mathcal{D}}(4) \oplus \left(\wedge^{k-1} N(\sigma) \otimes_{\mathbb{Z}} H_1(Y)\right) \oplus H_k(N^{\mathcal{D}}).$$

(2) *If $\mathcal{D}$ have complete locus, then $H_k(\widetilde{X}(\mathcal{D}))$ is isomorphic to*

$$H_k(X(\sigma) \times Y).$$

Lemma 3.3.2. *Let N, N', N'' be free finitely generated $\mathbb{Z}$-modules. Then the following statements hold.*

(1) *Let $f \colon N' \to N$ be a surjective homomorphism with kernel K. Then the kernel of $\wedge^k f \colon \wedge^k N' \to \wedge^k N$ is isomorphic to*

$$\bigoplus_{\substack{i+j=k \\ i \neq 0}} \wedge^i K \otimes_{\mathbb{Z}} \wedge^j N.$$

(2) *Let $K_1, \dots, K_r$ be submodules of N' and let $\mathcal{K}_i := \langle K_{i+1}, \dots, K_r \rangle$. Then the kernel of the sum homomorphism $K_1 \oplus \cdots \oplus K_r \to M$ is isomorphic to*

$$\bigoplus_{i=1}^{r-1} K_i \cap \mathcal{K}_i.$$

(3) *Let $f\colon N' \to N$ be a surjective homomorphism with kernel K and $i\colon N' \to N''$ be a homomorphism. Then the cokernel of the induced homomorphism $(f,i)\colon N' \to N \oplus N''$ is isomorphic to*

$$N''/i(K).$$

Proof of Proposition 3.3.1. We prove the first assertion. Recall that $N'(\sigma) \simeq N(\sigma) \oplus \mathbb{Z}$, then for each k we have

$$\wedge^k N'(\sigma) \simeq \wedge^k N(\sigma) \oplus \wedge^{k-1} N(\sigma).$$

Using [**16**, Proposition 12.3.1] we see that

$$H_k(U \cap U_{\mathcal{D}}) \simeq \oplus_{i=1}^r \wedge^k N'(\sigma) \simeq \wedge^k N(\sigma) \otimes_{\mathbb{Z}} \mathbb{Z}^r \oplus \wedge^{k-1} N(\sigma) \otimes_{\mathbb{Z}} \mathbb{Z}^r,$$

$$H_k(U^m) \simeq \oplus_{i=1}^r \wedge^k N(\Delta_i),$$

$$H_k(U_{\mathcal{D}}) \simeq \wedge^k N(\sigma) \oplus \wedge^{k-1} N(\sigma) \otimes_{\mathbb{Z}} (H_1(Y) \oplus \mathbb{Z}^r).$$

In order to compute $H_k(X(\mathcal{D}))$ we will study the homomorphism

$$i_k\colon H_k(U \cap U_{\mathcal{D}}) \to H_k(U) \oplus H_k(U_{\mathcal{D}}).$$

We denote by $i_{k,1}$ and $i_{k,2}$ the coordinates of i_k. Then we can write

$$i_{k,2}\colon \wedge^k N(\sigma) \otimes_{\mathbb{Z}} \mathbb{Z}^r \oplus \wedge^{k-1} N(\sigma) \otimes_{\mathbb{Z}} \mathbb{Z}^r \to \wedge^k N(\sigma) \oplus \wedge^{k-1} N(\sigma) \otimes_{\mathbb{Z}} (H_1(Y) \oplus \mathbb{Z}^r),$$

$$i_{k,1}\colon \oplus_{i=1}^r \left(\wedge^k N(\sigma) \oplus \wedge^{k-1} N(\sigma)\right) \to \oplus_{i=1}^r \wedge^k N(\Delta_i),$$

where $i_{k,2}$ is induced by the sum homomorphisms $\mathbb{Z}^r \to \mathbb{Z}$ and the injection $\mathbb{Z}^r \to H_1(Y) \oplus \mathbb{Z}^r$ and $i_{k,1}$ corresponds on each factor to the k-th wedge homomorphism of the projection $N(\sigma) \to N(\Delta_i)$ and the zero homomorphism on $\wedge^{k-1} N(\sigma)$.

Then, the kernel of i_k corresponds to the elements of $\oplus_{i=1}^r \wedge^k N(\sigma)$ which are in the kernel of the surjection $\oplus_{i=1}^r \wedge^k N(\sigma) \to \oplus_{i=1}^r \wedge^k N(\Delta_i)$ and in the kernel of the sum homomorphism $\oplus_{i=1}^r \wedge^k N(\sigma) \to \wedge^k N(\sigma)$. Using part (1) and (2) of Lemma 3.3.2 we conclude that the kernel of i_k is isomorphic to

$$\bigoplus_{i=1}^{r-1} \left(\bigoplus_{(a_1,a_2,a_3,a_4) \in \mathcal{I}_k} \wedge^{a_1,a_2,a_3,a_4} N_i^{\mathcal{D}} \right),$$

Observe that this is a free finitely generated abelian group.

On the other hand, $i_{k,1}$ is a surjection, then using part (1) and (3) of Lemma 3.3.2 we conclude that the cokernel of i_k is isomorphic to

$$\wedge^k N_1^{\mathcal{D}}(4) \oplus \left(\wedge^{k-1} N(\sigma) \otimes_{\mathbb{Z}} H_1(Y)\right)$$

Then the first assertion follows.

For the second assertion observe that as $\mathcal{D}$ have complete locus, then by [**5**, Lemma 18] we have that $N(\Delta_i) = N(\sigma)$ for each i. Then we have that $N_i^{\mathcal{D}}(1) = N_i^{\mathcal{D}}(2) = N_i^{\mathcal{D}}(3) = 0$ and $N_i^{\mathcal{D}}(4) = N(\sigma)$ for each i. Then the result follows from the first part. □

Now we turn to compute the cohomology of affine complexity-one $\mathbb{T}$-varieties. Given a cone $\sigma \subseteq N$ we denote by $M(\sigma) := \sigma^{\perp} \cap M$. We recall from [**16**, Proposition 12.3.1] that $H^k(X(\sigma)) \simeq \wedge^k M(\sigma)$ for each k. Then, given a p-divisor $\mathcal{D} := \sum_{i=1}^r \Delta_i \oplus p_i$ on a semiprojective curve Y. Observe that for each i we have isomorphisms $M(\sigma) := M(\Delta_i) \oplus M_{\Delta_i}$ and $M'(\sigma_i) \simeq M(\Delta_i)$, where $M_{\Delta_i} := (N_{\Delta_i})^{\perp}$ for each i.

We will use the following notation

$$\begin{array}{rcl} M_i^{\mathcal{D}}(1) & := & M_{\Delta_{i+1}} \cap \langle M(\Delta_1), \dots, M(\Delta_i) \rangle, \\ M_i^{\mathcal{D}}(2) & := & M_{\Delta_{i+1}} \cap M(\Delta_1) \cap \dots \cap M(\Delta_i), \\ M_i^{\mathcal{D}}(3) & := & M(\Delta_{i+1}) \cap \langle M(\Delta_1), \dots, M(\Delta_i) \rangle, \\ M_i^{\mathcal{D}}(4) & := & M(\Delta_1) \cap \dots \cap M(\Delta_{i+1}). \end{array}$$

Observe that the subspaces $M^{\mathcal{D}}(1)M^{\mathcal{D}}(2), M^{\mathcal{D}}(3)$ and $M^{\mathcal{D}}(4)$ are pairwise disjoint and generate $M(\sigma)$. To abreviate notation, given natural numbers a_1, a_2, a_3, a_4 we will write

$$\wedge^{a_1,a_2,a_3,a_4} H_k(M^{\mathcal{D}}) := \wedge^{a_1} M_i^{\mathcal{D}}(1) \otimes_{\mathbb{Z}} \wedge^{a_2} M_i^{\mathcal{D}}(2) \otimes_{\mathbb{Z}} \wedge^{a_3} M_i^{\mathcal{D}}(3) \otimes_{\mathbb{Z}} \wedge^{a_4} M_i^{\mathcal{D}}(4),$$

and

$$H^k(M^{\mathcal{D}}) := \bigoplus_{i=1}^{r-1} \left(\bigoplus_{(a_1,a_2,a_3,a_4) \in \mathcal{I}_{k-1}} \wedge^{a_1,a_2,a_3,a_4} M_i^{\mathcal{D}} \right).$$

PROPOSITION 3.3.3. *Let $\mathcal{D} := \sum_{i=1}^r \Delta_{p_i} \otimes p_i$ be a p-divisor on a curve Y. Then the following statements holds.*

- *If $\mathcal{D}$ have affine locus then $H^k(\widetilde{X}(\mathcal{D}))$ is isomorphic to*
$$\wedge^k M^{\mathcal{D}}(4) \oplus \left(\wedge^{k-1} M(\sigma) \otimes_{\mathbb{Z}} H^1(Y)\right) \oplus H^k(M^{\mathcal{D}}).$$
- *If $\mathcal{D}$ have complete locus then $H^k(\widetilde{X}(\mathcal{D}))$ is isomorphic to*
$$H^k(X(\sigma) \times Y) \oplus H^k(X(\sigma)).$$

LEMMA 3.3.4. *Let M be a free finitely generated $\mathbb{Z}$-module, $K_1, \dots, K_r$ be submodules and $\mathcal{K}^i := K_1 \cap \dots \cap K_i$. Then the cokernel of the homomorphism*

$$\bigoplus_{i=1}^r K_i \oplus M \to M^r,$$

given by $(k_1, \dots, k_r, m) \to (m - k_1, \dots, m - k_r)$ is isomorphic to

$$\bigoplus_{i=1}^{r-1} M / \langle \mathcal{K}^i, K_{i+1} \rangle.$$

PROOF OF PROPOSITION 3.3.3. We proof the first assertion. Recall that $M'(\sigma) \simeq M(\sigma) \oplus \mathbb{Z}$, then for each k we have

$$\wedge^k M'(\sigma) \simeq \wedge^k M(\sigma) \oplus \wedge^{k-1} M(\sigma).$$

Using [16, Proposition 12.3.1] we see that

$$H^k(U \cap U_{\mathcal{D}}) \simeq \oplus_{i=1}^r \wedge^k M'(\sigma) \simeq \wedge^k M(\sigma) \otimes_{\mathbb{Z}} \mathbb{Z}^r \oplus \wedge^{k-1} M(\sigma) \otimes_{\mathbb{Z}} \mathbb{Z}^r,$$

$$H^k(U) \simeq \oplus_{i=1}^r \wedge^k M(\Delta_i),$$

$$H_k(U_{\mathcal{D}}) \simeq \wedge^k M(\sigma) \oplus \wedge^{k-1} M(\sigma) \otimes_{\mathbb{Z}} (H^1(Y) \oplus \mathbb{Z}^r).$$

In order to compute $H^k(X(\mathcal{D}))$ we will study the homomorphism

$$i^k \colon H^k(U) \oplus H^k(U_{\mathcal{D}}) \to H^k(U \cap U_{\mathcal{D}}).$$

We denote by $i^{k,1}$ and $i^{k,2}$ the homomorphism i^k restricted to $H^k(U)$ and $H^k(U_{\mathcal{D}})$ respectively. Then we can write

$$i^{k,1}\colon \bigoplus_{i=1}^{r} \wedge^k M(\Delta_i) \to \bigoplus_{i=1}^{r} \left(\wedge^k M(\sigma) \oplus \wedge^{k-1} M(\sigma)\right),$$

$$i^{k,2}\colon \wedge^k M(\sigma) \oplus \wedge^{k-1} M(\sigma) \otimes_{\mathbb{Z}} (H^1(Y) \oplus \mathbb{Z}^r) \to \wedge^k M(\sigma) \otimes_{\mathbb{Z}} \mathbb{Z}^r \oplus \wedge^{k-1} M(\sigma) \otimes_{\mathbb{Z}} \mathbb{Z}^r,$$

where $i^{k,1}$ is the k-th wedge product of the inclusion $M(\Delta_i) \to M(\sigma)$ on each component and $i^{k,2}$ is the homomorphism induced by $\mathbb{Z} \to \mathbb{Z}^r, z \mapsto (z, \dots, z)$ and the projection $H^1(Y) \oplus \mathbb{Z}^r \to \mathbb{Z}^r$.

Then the kernel of i^k is isomorphic to $\wedge^{k-1} M(\sigma) \otimes_{\mathbb{Z}} H^1(Y)$ direct sum with the kernel of the homomorphism

$$\bigoplus_{i=1}^{r} \wedge^k M(\Delta_i) \oplus \wedge^k M(\sigma) \to (\wedge^k M(\sigma))^r, \tag{3.3.1}$$

which is isomorphic to $\left(\wedge^k (M(\Delta_1) \cap \dots \cap M(\Delta_r))\right)^{r-1}$. Observe that the kernel of i^k is a free finitely generated abelian group. On the other hand, the cokernel of i^k is the cokernel of homomorphism 3.3.1. Then, using Lemma 3.3.4 we see that the cokernel of i^k is isomorphic to

$$\bigoplus_{i=1}^{r-1} \left(\bigoplus_{(a_1,a_2,a_3,a_4) \in \mathcal{I}_{k-1}} \wedge^{a_1,a_2,a_3,a_4} M_i^{\mathcal{D}} \right),$$

Then the first assertion follows.

For the second assertion observe that as $\mathcal{D}$ have complete locus, then by [**5**, Lemma 18] we have that $M(\Delta_i) = M(\sigma)$ for each i. Then we have that $M_i^{\mathcal{D}}(1) = M_i^{\mathcal{D}}(2) = M_i^{\mathcal{D}}(3) = 0$ and $M^{\mathcal{D}}(4) = M(\sigma)$ for each i. Then the result follows from the first part. □

REMARK 3.3.5. Proposition 3.3.3 can be obtained from Proposition 3.3.1 and the Universal coefficient theorem for cohomology. In this case using the Universal coefficient theorem correspond to take duality $N(\sigma) \mapsto M(\sigma)$.

Let $\mathcal{S}$ be a divisorial fan on a curve Y, we will denote by I an ordered set of indexes i such that $\mathcal{D}^i \in \mathcal{S}$ is a maximal p-divisor with respect to the inclusion. We denote by I^k the set of increasing k-sequences of elmenets of I.

PROPOSITION 3.3.6. *Let $\mathcal{S}$ be a divisorial fan on a curve Y such that for each $i \in I$, $\mathcal{D}^i$ has affine locus and full-dimensional tailcone. Then*

$$H^2(\widetilde{X}(\mathcal{S})) \simeq \ker \left(\bigoplus_{(i,j) \in I^2} M^{\mathcal{D}^i \cap \mathcal{D}^j}(4) \to \bigoplus_{(i,j,k) \in I^3} M^{\mathcal{D}^i \cap \mathcal{D}^j \cap \mathcal{D}^k}(4). \right) \oplus H^2(\mathrm{Loc}(\mathcal{S})),$$

where the homomorphism $M^{\mathcal{D}^i \cap \mathcal{D}^j}(4) \to M^{\mathcal{D}^i \cap \mathcal{D}^j \cap \mathcal{D}^k}(4)$, is induced by the inclusion of p-divisors $\mathcal{D}^i \cap \mathcal{D}^j \cap \mathcal{D}^k \subseteq \mathcal{D}^i \cap \mathcal{D}^j$.

PROOF. Consider the affine open cover of $X(\mathcal{S})$ given by $\mathcal{U} := \{X(\mathcal{D}^i) \mid i \in I\}$. We have an spectral sequence of the covering $\mathcal{U}$

$$E_1^{p,q} = \bigoplus_{\gamma = (i_0, \dots, i_p) \in I^{p+1}} H^q(X(\mathcal{D}^{i_0}) \cap \dots \cap X(\mathcal{D}^{i_p})) \Rightarrow H^{p+q}(X(\mathcal{S})).$$

Observe that

$$E_1^{p,0} = \bigoplus_{\gamma=(i_0,\dots,i_p)\in I^{p+1}} \mathbb{Z}.$$

Thus, $E_2^{p,0} = 0$ for $p > 0$ and $E_2^{0,0} = \mathbb{Z}$. The maximal cones of $\Sigma(\mathcal{S})$ are full-dimensional, then by Proposition 3.3.3 we have that $E_1^{0,q} = \bigoplus_{i\in I} H^q(X(\mathcal{D}^i)) = 0$, for all $q > 1$ and $E_1^{0,1} = \bigoplus_{i\in I} H^1(\mathrm{Loc}(\mathcal{D}^i))$. It follows that $E_2^{0,q} = 0$ for all $q > 1$. Then $E_r^{2,0}$ and $E_r^{0,2}$ are zero for all $r \geq 2$. Moreover, the differentials into and out of $E_r^{1,1}$ are zero for all $r \geq 2$. Thus, $E_2^{1,1} = E_\infty^{1,1} \simeq H^2(X(\mathcal{S}))$. Then, we can compute $H^2(X(\mathcal{S}))$ from the complex

$$E_1^{0,1} \to E_1^{1,1} \to E_1^{2,1}.$$

By Proposition 3.3.1 we have that $H^2(X(\mathcal{S}))$ is the homology of the following complex

$$\bigoplus_{i\in I} H^1(\mathrm{Loc}(\mathcal{D}^i)) \to \bigoplus_{(i,j)\in I^2} M^{\mathcal{D}^i\cap\mathcal{D}^j}(4) \oplus H^1(\mathrm{Loc}(\mathcal{D}^i\cap\mathcal{D}^j))$$
$$\to \bigoplus_{(i,j,k)\in I^3} M^{\mathcal{D}^i\cap\mathcal{D}^j\cap\mathcal{D}^k}(4) \oplus H^1(\mathrm{Loc}(\mathcal{D}^i\cap\mathcal{D}^j\cap\mathcal{D}^k)).$$

Observe that this group is isomorphic to

$$\ker\left(\bigoplus_{(i,j)\in I^2} M^{\mathcal{D}^i\cap\mathcal{D}^j}(4) \to \bigoplus_{(i,j,k)\in I^3} M^{\mathcal{D}^i\cap\mathcal{D}^j\cap\mathcal{D}^k}(4)\right),$$

direct sum with the homology of the complex

$$\bigoplus_{i\in I} H^1(\mathrm{Loc}(\mathcal{D}^i)) \to \bigoplus_{(i,j)\in I^2} H^1(\mathrm{Loc}(\mathcal{D}^i\cap\mathcal{D}^j)) \to \bigoplus_{(i,j,k)\in I^3} H^1(\mathrm{Loc}(\mathcal{D}^i\cap\mathcal{D}^j\cap\mathcal{D}^k)),$$

which is isomorphic to $H^2(\mathrm{Loc}(\mathcal{S}))$. □

In the following example we use Theorem 3 and Proposition 3.3.1 to compute the cohomology groups of a particular affine $\mathbb{T}$-variety.

Example 3.3.7. Consider a divisorial fan $\mathcal{S}$ on $(Y, \mathbb{Z}^3)$ with Y a curve. Let $\{y_1, y_2\} \subseteq Y$ be the support of $\mathcal{S}$. Assume that the fan of the generic fiber is generated by e_1 and the slices $\mathcal{S}_1$ and $\mathcal{S}_2$ over y_1 and y_2 respectively corresponds to the polyhedra

$$\Delta_1 := \overline{(0,0,0),(0,1,0)} + e_1, \qquad \Delta_2 := \overline{(0,0,0),(0,0,1)} + e_1.$$

Using Theorem 3 we observe that $\pi_1(\widetilde{X}(\mathcal{S})) \simeq \pi_1(Y)$, then $H_1(\widetilde{X}(\mathcal{S})) \simeq H_1(Y)$. Moreover, using Proposition 3.3.1 we can compute

$$N_1^{\mathcal{D}}(1) = \{0\}, \qquad N_1^{\mathcal{D}}(2) = \langle e_2\rangle \qquad N_1^{\mathcal{D}}(3) = \langle e_3\rangle, \qquad N_1^{\mathcal{D}}(4) = 0,$$
$$H_3(N^{\mathcal{D}}) \simeq \mathbb{Z}, \qquad \text{and 0 otherwise}.$$

Then, we conclude that

$$H_0(\widetilde{X}(\mathcal{S})) \simeq \mathbb{Z}, \qquad H_1(\widetilde{X}(\mathcal{S})) \simeq H_1(Y), \qquad H_2(\widetilde{X}(\mathcal{S})) \simeq H_1(Y)\otimes_{\mathbb{Z}} \mathbb{Z}^2,$$
$$H_3(\widetilde{X}(\mathcal{S})) \simeq H_1(Y)\oplus\mathbb{Z}, \qquad H_k(\widetilde{X}(\mathcal{S})) \simeq 0, \text{ for } k \geq 4.$$

Future directions

In this section we discuss future directions of the work in Chapter 2 and Chapter 3. Given an algebraic variety X, The problem of calculating the dimension of a linear system of hypersurfaces of X of a fixed degree passing through points with prescribed multiplicities is a wide open problem. In particular, the problem is still open in $\mathbb{P}^2$ and most of results are for bounded degree, bounded multiplicities or a fixed number of points. However, there are works which try to describe the subvarieties appearing in the base locus of a linear system in a product of projective spaces (see [**8**,**9**,**44**]). Toric varieties seem to be good candidates to generalize these results. This gives the motivation for the following problems.

Problem 3.3.8. Let X be a projective smooth toric variety and let $p_1, \ldots, p_r \in X$ be points in very general position. $\mathcal{L}$ be the linear system of hypersurfaces of degree $[D] \in \mathrm{Cl}(X)$ passing through p_i with multiplicity m_i.

- What are the subvarieties which appear in $\mathrm{Bs}(\mathcal{L})$. Even new examples would be interesting.
- Write a conjecture which try to predict $\dim(\mathcal{L})$. Even new lower bounds would be interesting in any case.
- Find new examples of toric varieties X and points $p_1, \ldots, p_r \in X$ such that the blow-up along these points is a Mori dream space.

The results presented in Chapter 3 depends in the singularities of the $\mathbb{T}$-variety X. In [**35**] there are characterisations of singularities of $\mathbb{T}$-varieties, they focus to give a complete picture of the singularities of complexity-one $\mathbb{T}$-varieties and its descriptions via the divisorial fan. In order to have a better understanding of the topology of a $\mathbb{T}$-variety its necessary the study of its singularities. This motivates the following problems.

Problem 3.3.9. Let X be an affine $\mathbb{T}$-variety, with its unique fixed point p, and let R be the ring of regular functions of X and m the maximal ideal of R corresponding to p.

- Study the local cohomology groups $H^p_m(R)$, for $p \geq 1$. This is open also in the complexity-one case.
- Let $\mathcal{D}$ be the p-divisor of X, try to find a characterisation of the Cohen-Macaulayness of X via $\mathcal{D}$. This is open also in the complexity-one case.

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