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Minimización del primer valor propio del p-Laplaciano de Dirichlet en cierto tipo de dominios.

(Minimization of the first eigenvalue of the Dirichlet p-Laplacian in certain classes of domains.)

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Esta tesis esta dedicada a María Ester mi madre, Felipe y Pablo, mis hermanos y a la memoria de Juan Luis, mi difunto padre.

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Introduction.

In this thesis, we will work around the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Δ_p is the p- Laplacian operator, with $1 < p < \infty$, which is a generalization of the Laplacian operator ($p = 2$) and it is defined for a function u in the Sobolev space $W_0^{1,p}(\Omega)$ as

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

More specifically, we will study thoroughly the first eigenvalue $\lambda_1(\Omega)$ of p- Laplacian with Dirichlet condition, which is defined as the minimum of Rayleigh quotient for nonzero functions belonging to $W_0^{1,p}(\Omega)$. i.e.,

$$\lambda_1(\Omega) = \min_{\varphi \in W_0^{1,p}(\Omega), \phi \neq 0} \frac{\int_\Omega |\nabla \varphi|^p}{\int_\Omega |\varphi|^p}.$$

We note that , λ_1 depends on the domain Ω. We will show the principal properties of $\lambda_1(\Omega)$ and of its eigenfunctions, and later obtain results on the problem of minimization of $\lambda_1(\Omega)$ in certain classes of domains with the same volume or perimeter, similar to a classical problem.

In the first chapter, which corresponds to the preliminaries, we will introduce some basic notions and definitions. We introduce the notion of a distribution, which allows us to define the concept of weak derivative of a function defined in a domain Ω, among other notions. Moreover, in the first chapter we will define the Sobolev space $W^{1,p}(\Omega)$, which is the set of all functions which belong to $L^p(\Omega)$, such that all its weak derivatives of first order also belong to $L^p(\Omega)$. Our aim in this chapter is to recall the definition and to study certain properties of the space $W_0^{1,p}(\Omega)$, which is the domain of the operator p- Laplacian. We also recall some useful notions and results from measure theory and recall the notion of Hausdorff convergence of sets.

In the second chapter, we will describe the principal tools which will allow us to solve the problem of minimization of λ_1 in certain classes of domains. We will introduce the Schwarz symmetrization and the Steiner symmetrization, and discuss their main properties.

In the third chapter, we will describe the principal properties of $\lambda_1(\Omega)$, which we will use for solving the minimization problems previously mentioned. We will see that $\lambda_1(\Omega)$ is a simple eigenvalue, with eigenfunctions which are either strictly positive or strictly negative on Ω and these belong to $C^1(\overline{\Omega})$. Also we will see that $\lambda_1(\Omega)$ is invariant by translation and it varies only by a constant if we apply a homothety to the domain Ω, among other properties. Further, we will show that $\lambda_1(\Omega)$ as a function of the domain is continuous with respect to the topology induced by the Hausdorff distance in certain class of domains. Another important tool is the differentiability of λ_1 with respect to domain variations, which will allow us in the last chapter to develop an alternative proof of an analogue of the Faber- Krahn inequality for the p- Laplacian, which says that the domain that minimizes $\lambda_1(\Omega)$ among domains of the same volume with C^2 boundary is a sphere.

Finally, in the fourth and last chapter we prove the Faber-Krahn inequality for the p- Laplacian and discuss two proofs of the uniqueness of the ball as a minimizer, one being classical and another being an alternative proof. We will see that the classical proof uses Schwarz symmetrization and that the alternative proof uses the fact that if Ω is a minimal domain for λ_1 as a function of the domain, among domains with the same area, then the eigenfunction corresponding to $\lambda_1(\Omega)$ has the property that its normal derivative is constant on $\partial\Omega$. We will show that the only domain that satisfies this condition is a sphere. As a corollary, we will prove an analogous result for domains with a given surface area. Moreover, we will study the minimization of $\lambda_1(\Omega)$ among triangular and quadrilateral planar domains , obtaining some generalizations of known results for the Dirichlet Laplacian. Here, the continuity of λ_1 as a function of the domain and the Steiner symmetrization acquire a relevant role.

Francisco Toledo OÃśate, December 2012.

Chapter 1

Preliminaries

In this chapter, we will fix some notations, will give some basic definitions and will recall some theorems which shall be employed later on. The treatment of this chapter is based principally on Kesavan [18].

1.1 Distributions

Definition 1.1.1. *Let φ a real valued continuous function defined on an open set Ω of $\mathbb{R}^n$. The* ***support*** *of φ, written as supp(φ), is defined as:*

$$\overline{\{x \in \Omega : \varphi(x) \neq 0\}} \tag{1.1.1}$$

with the closure considered in $\mathbb{R}^n$.

If this closed set is compact as well, then φ is said to be of compact support. The set of all infinitely differentiable (i.e. C^∞) functions defined on $\mathbb{R}^n$ with compact support is a vector space which will henceforth denoted by $D(\mathbb{R}^n)$ or, simply, D. This vector space is said the space of test functions. If Ω is any open set in $\mathbb{R}^n$, we can still talk of the space of C^∞ functions with compact support, the support being contained in Ω. This space will be denoted by $D(\Omega)$. We can consider a topology in $D(\Omega)$ which will make it a topological vector space (See Appendix 2 in [18]). However, we will just define convergent sequences in $D(\Omega)$.

Definition 1.1.2. *A sequence of functions* $\{\varphi_m\}_{m=1}^{\infty}$ *in* $D(\Omega)$ *is said to* ***converge to*** 0 *if there exists a fixed compact set* $K \subset \Omega$ *such that* $supp(\varphi_m) \subset K$ *for all* m*,* φ_m *and all its derivatives converge uniformly to zero on* K*.*

Definition 1.1.3. *A linear functional* $T : D(\Omega) \to \mathbb{R}$*, is said to be a* ***distribution*** *on* Ω *if whenever* $\varphi_m \to 0$ *in* $D(\Omega)$*, we have* $T(\varphi_m) \to 0$*.*

The space of distributions, which is the dual of the space of test functions, is denoted by $D'(\Omega)$.

Example 1.1.4. *A function* $f : \Omega \to \mathbb{R}$ *is said to be* ***locally integrable*** *if for every compact set* $K \subset \Omega$*,*

$$\int_K |\ f| < +\infty.$$

For instance, a continuous function in Ω *is locally integrable. Given a locally integrable function* f *in* Ω*,* $T_f : D(\Omega) \to \mathbb{R}$ *defined by*

$$T_f(\varphi) = \int_\Omega f\varphi. \tag{1.1.2}$$

is a distribution. This follows from the estimate

$$|T_f(\varphi)| \leq \left(\int_K |f| \right) \|\varphi\|_\infty$$

where K *is a compact subset of* Ω *such that* $supp(\varphi) \subset K$*.*

Given a distribution T, we say "T is a function" to mean that there exists a locally integrable function f such that $T = T_f$. We denote the distribution T_f by f. It is important to say that there exists distributions which cannot be generated by a locally integrable function, for instance, the *Dirac* distribution (see [18], pag.7).

Now we introduce the concept of differentiation of distributions.

Definition 1.1.5. *Let $T \in D'(\Omega)$. The i^{th}* ***partial derivative of*** *T is a distribution $\dfrac{\partial T}{\partial x_i}$ defined by*

$$\frac{\partial T}{\partial x_i}(\varphi) = -T\left(\frac{\partial \varphi}{\partial x_i}\right) \quad \forall \varphi \in D(\Omega).$$

Definition 1.1.6. *If f, g are locally integrable functions and*

$$\frac{\partial T_f}{\partial x_i}(\varphi) = T_g(\varphi) \quad \forall \varphi \in D(\Omega)$$

then we will say that g is the i^{th} ***weak (or distributional) partial derivative of*** *f. We note $\dfrac{\partial T_f}{\partial x_i}$ by $\dfrac{\partial f}{\partial x_i}$.*

Remark 1.1.7. *If $\Omega \subset \mathbb{R}^n$ and f belongs to $C^1(\Omega)$, then the i^{th} weak partial derivative of f coincides with the classical i^{th} partial derivative of f.*

1.2 Sobolev Spaces

Let Ω be an open set in $\mathbb{R}^n$.

Definition 1.2.1. *Let $1 \leq p < \infty$. The* ***Sobolev space*** *$W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega) \text{ for all } i = 1, 2, \ldots, n \right\}.$$

In other words, $W^{1,p}(\Omega)$ is the collection of all functions in $L^p(\Omega)$ such that all its weak partial derivatives of the first order are also in $L^p(\Omega)$. $W^{1,p}(\Omega)$ is a normed vector space provided with the norm:

$$\|u\|_{1,p,\Omega} = \|u\|_p + \|\nabla u\|_p \tag{1.2.1}$$

Remark 1.2.2. *It will also be convenient to use the semi $-$ norm*

$$|u|_{1,p,\Omega} = \|\nabla u\|_p. \tag{1.2.2}$$

We note that the map

$$u \in W^{1,p}(\Omega) \to \left(u, \frac{\partial u}{\partial x_1}, \dots \frac{\partial u}{\partial x_n}\right) \in (L^p(\Omega))^{n+1} \tag{1.2.3}$$

is an isometry of $W^{1,p}(\Omega)$ into $(L^p(\Omega))^{n+1}$, if we provide the latter space with the norm

$$\|f\| = \|f_0\|_p + \|(f_1, f_2, \dots, f_n)\|_p$$

for $f = (f_i)_{i=0}^n \in (L^p(\Omega))^{n+1}$. This is a useful fact to remember and will be used in the proof of the following result.

Theorem 1.2.3. *For every $1 \le p < \infty$, the space $W^{1,p}(\Omega)$ is a separable Banach space. If $1 < p < \infty$, it also is a reflexive space.*

Proof: We first show that $W^{1,p}(\Omega)$ is a Banach space. Let u_m be a Cauchy sequence in $W^{1,p}(\Omega)$. It follows from the definition of the norm that $\{u_m\}$ and $\{\frac{\partial u_m}{\partial x_i}\}$, $1 \le i \le n$, are Cauchy sequences in $L^p(\Omega)$. As $L^p(\Omega)$ is Banach, then $u_m \to u$ and $\frac{\partial u_m}{\partial x_i} \to v_i, 1 \le i \le n$, with u and v_i in $L^p(\Omega)$. The completeness of the $W^{1,p}(\Omega)$ will be proved if we show that $\frac{\partial u}{\partial x_i} = v_i$ in the sense of distributions.

Let $\varphi \in D(\Omega)$. We need to show that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega v_i \varphi$$

because this implies that

$$\left(\frac{\partial T_u}{\partial x_i}\right)(\varphi) = -T_u\left(\frac{\partial \varphi}{\partial x_i}\right) = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} = \int_\Omega v_i \varphi = T_{v_i}(\varphi).$$

Now, since $u_m \in W^{1,p}(\Omega)$, by the definition of weak derivative we know that

$$\int_\Omega u_m \frac{\partial \varphi}{\partial x_i} = -\int_\Omega \frac{\partial u_m}{\partial x_i} \varphi.$$

Further, since $\varphi \in D(\Omega)$, we have that φ and $\frac{\partial \varphi}{\partial x_i}$ are in $L^q(\Omega)$ for all i and for all q, in particular, the conjugate exponent $q = p/(p-1)$. Thus,

$$\int_\Omega \left|(u_m - u)\frac{\partial \varphi}{\partial x_i}\right| \le \|u_m - u\|_p \left\|\frac{\partial \varphi}{\partial x_i}\right\|_q \to 0,$$

$$\int_\Omega \left| \left(\frac{\partial u_m}{\partial x_i} - v_i \right) \varphi \right| \leq \left\| \frac{\partial u_m}{\partial x_i} - v_i \right\|_p \|\varphi\|_q \to 0.$$

This implies that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = - \int_\Omega v_i \varphi.$$

We note that $(L^p(\Omega))^{n+1}$ is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$ (it is a cartesian product of reflexive and separable spaces respectively). Since $W^{1,p}(\Omega)$ is complete, its image under the isometry (1.2.3) is a closed subspace of $(L^p(\Omega))^{n+1}$. Therefore, $W^{1,p}(\Omega)$ inherits the corresponding properties of $(L^p(\Omega))^{n+1}$. □

We introduce an important subspace of $W^{1,p}(\Omega)$. If $1 \leq p < \infty$, we know that $D(\Omega)$ is dense in $L^p(\Omega)$. Also, $D(\Omega) \subset W^{1,p}(\Omega)$, for $1 \leq p \leq \infty$. Thus we have the next definition.

Definition 1.2.4. *If $1 \leq p < \infty$, we define $W_0^{1,p}(\Omega)$ as the closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.*

The following theorem will allow us to conclude that if $u \in W_0^{1,p}(\Omega)$, then $|u| \in W_0^{1,p}(\Omega)$.

Theorem 1.2.5. *(Stampacchia) Let G be a Lipschitz continuous function of $\mathbb{R}$ into itself such that $G(0) = 0$. Then if Ω is bounded, $1 < p < \infty$ and $u \in W_0^{1,p}(\Omega)$ we have $G \circ u \in W_0^{1,p}(\Omega)$.*

Proof: See page 60 [18]. □

Theorem 1.2.6. *(Poincare's inequality) Let Ω be a bounded open set in $\mathbb{R}^n$. Then there exists a postive constant $C = C(\Omega, p)$ such that*

$$|u|_{0,p,\Omega} \leq C|u|_{1,p,\Omega} \quad \textit{for every } u \in W_0^{1,p}(\Omega) \tag{1.2.4}$$

In particular, the seminorm defined in (1.2.2) is a norm on $W_0^{1,p}(\Omega)$, which is equivalent to the norm $\|\cdot\|_{1,p,\Omega}$ on $W_0^{1,p}(\Omega)$.

Proof: Let $\Omega = (-a,a)^n$, $a > 0$. Let $u \in D(\Omega)$. Then

$$u(x) = \int_{-a}^{x_n} \frac{\partial u}{\partial x_n}(x',t)\ dt,\ x = (x',x_n),\ \text{since}\ u(x',-a) = 0.$$

By Holder's inequality,

$$|u(x)| \leq \left(\int_{-a}^{x_n} \left|\frac{\partial u}{\partial x_n}(x',t)\right|^p\ dt\right)^{\frac{1}{p}} |x_n + a|^{\frac{1}{q}},$$

whenever p, q are conjugate exponent. Hence,

$$|u(x)|^p \leq |x_n + a|^{\frac{p}{q}} \int_{-a}^{x_n} \left|\frac{\partial u}{\partial x_n}(x',t)\right|^p\ dt.$$

Integrating over x' and since $x_n \leq a$,

$$\int |u(x',x_n)|^p\ dx' \leq (2a)^{\frac{p}{q}} \int_\Omega \left|\frac{\partial u}{\partial x_n}\right|^p,$$

next integrating over x_n,

$$\int_\Omega |u(x)|^p\ dx \leq (2a)^{\frac{p}{q}+1} \int_\Omega \left|\frac{\partial u}{\partial x_n}\right|^p \leq (2a)^{\frac{p}{q}+1} \int_\Omega |\nabla u(x)|^p\ dx.$$

Thus

$$|u|_{0,p,\Omega} \leq C|u|_{1,p,\Omega}$$

with $C = (2a)^{\frac{p}{q}+1}$. This proves (1.2.4) for $u \in D(\Omega)$. But as $D(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ and both sides of inequality (1.2.4) are continuous in u for the topology in $W_0^{1,p}(\Omega)$), the inequality follows for all $u \in W_0^{1,p}(\Omega)$. If Ω is not a "box" let $\hat{\Omega}$ box of the form $(-a,a)^n$ such that $\Omega \subset \hat{\Omega}$, and extend $u \in W_0^{1,p}(\Omega)$ by zero to get $\hat{u} \in W_0^{1,p}(\hat{\Omega})$. Finally apply (1.2.4) which is available for $\hat{\Omega}$. □

The following two theorems will be used in the future, but we omit their proofs.

Theorem 1.2.7. *Let* $1 \leq p < \infty$ *and* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. *If* $u = 0$ *on* $\partial\Omega$, *then* $u \in W_0^{1,p}(\Omega)$.

Proof: See page 61 [18]. □

Theorem 1.2.8. *(Rellich- Kondrasov) Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain of* C^1 *boundary. Then the following inclusions are compact.*

1. *if* $p < n$, $W_0^{1,p}(\Omega) \to L^q(\Omega)$, $1 \leq q < \frac{np}{n-p}$
2. *if* $p = n$, $W_0^{1,n}(\Omega) \to L^q(\Omega)$, $1 \leq q < \infty$
3. *if* $p > n$, $W_0^{1,p}(\Omega) \to C(\overline{\Omega})$.

Proof: See page 84 [18]. □

Definition 1.2.9. *Let* $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ *be a bounded domain and* $f \in W^{-1,q}(\Omega)$ *where* p, q *are conjugate exponents. We say that a function* $u \in W^{1,p}(\Omega)$ *is a weak solution of the problem*

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = f \text{ in } \Omega, \tag{1.2.5}$$

if

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \varphi \rangle = \int_\Omega f\varphi \;\forall \varphi \in W_0^{1,p}(\Omega). \tag{1.2.6}$$

1.3 Hausdorff distance

We define a distance between certain kinds of sets in $\mathbb{R}^n$.

Definition 1.3.1. *Let X, Y be two non-empty compacts sets in $\mathbb{R}^n$. We set*

$$\forall z \in \mathbb{R}^n, d(z, X) := \min_{x \in X} |x - z|,$$

$$\rho(X, Y) := \max_{x \in X} d(x, Y).$$

Then the ***Hausdorff distance*** *$d^H(X, Y)$ between X and Y is defined as*

$$d^H(X, Y) := \max(\rho(X, Y), \rho(Y, X)).$$

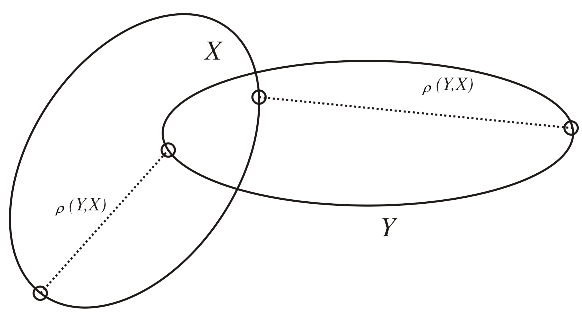


Figure 1.1: The Hausdorff distance between two compacts set X, Y is $\max(\rho(X, Y), \rho(Y, X))$.

For open subsets of a compact set D, we define the Hausdorff distance through their complement in D, which are compacts sets.

Definition 1.3.2. *Let X, Y be two open subsets of a compact D. Then the* ***Hausdorff distance*** *$d_H(X, Y)$ between X and Y is defined by*

$$d_H(X, Y) = d^H(D \setminus X, D \setminus Y).$$

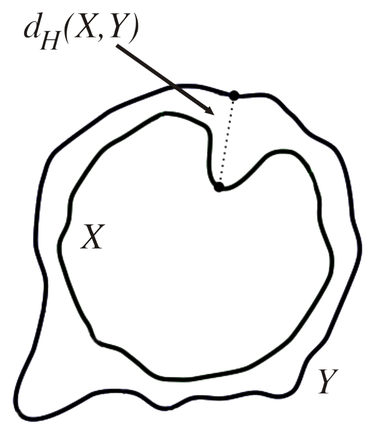


Figure 1.2: The Hausdorff distance between two open sets X, Y which are contained in a ball fixed D.

Roughly speaking, the Hausdorff distance between two open subsets of a compact set, is the distance between their boundaries. An useful property of the Hausdorff distance is the following:

Theorem 1.3.3. *Let D be a fixed compact set in $\mathbb{R}^n$ and $(\Omega_n)_{n=1}^{\infty}$ be a sequence of open subsets of D. Then, there exists an open set $\Omega \subset B$ and a subsequence (Ω_{n_k}) which converges for the Hausdorff distance to Ω.*

Proof: See [7] and [15]. □

1.4 Measure Theory

In this section we describe some measure theoretic results which will be used through out this work.

Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, denote its n- dimensional Lebesgue measure by $|E|$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Let $u : \Omega \to \mathbb{R}$ be a bounded measurable function. For $t \in \mathbb{R}$, the **level set** $\{u > t\}$ or Ω_t is defined as

$$\{u > t\} = \{x \in \Omega : u(x) > t\}. \tag{1.4.1}$$

The sets $\{u < t\}, \{u \geq t\}, \{u = t\}$ are defined analogously.

Definition 1.4.1. *Let $u : \Omega \to \mathbb{R}$ be a bounded measurable function and let $t \in \mathbb{R}$. The **distribution function** of u is given by*

$$\mu_u(t) = |\{u > t\}|.$$

Remark 1.4.2. *1. The function μ_u is nonincreasing. In fact, whenever $t_1 \geq t_2$, if $x \in \{u > t_1\}$, then $u(x) > t_1 \geq t_2$, thus $x \in \{u > t_2\}$. Therefore $\{u > t_1\} \subset \{u > t_2\}$. It follows that*

$$\mu_u(t_1) = |\{u > t_1\}| \leq |\{u > t_2\}| = \mu_u(t_2).$$

2. Let $t < \inf u$. Since $u \geq \inf u$, we have $\{u > t\} = \Omega$, thus $\mu_u(t) = |\Omega|$.

3. Let $t \geq \sup u$. In this case $\{u > t\} = \emptyset$, thus $\mu_u(t) = 0$. Hence, the range of μ_u is contained in the interval $[0, |\Omega|]$.

4. The distribution function μ_u is right- continuous in $\mathbb{R}$. In fact, if $t \in \mathbb{R}$, then

$$0 \leq \mu_u(t) - \mu_u(t+h) = |\{u > t\}| - |\{u > t+h\}| = |\{t < u \leq t+h\}|.$$

We observe that $(\{t < u \leq t+h\})_{h>0}$ decreases as $h \to 0^+$, and $\bigcap_{h>0} \{t < u \leq t+h\} = \emptyset$, thus

$$\lim_{h \to 0^+} (\mu_u(t) - \mu_u(t+h)) = \lim_{h \to 0^+} |\{t < u \leq t+h\}| = 0.$$

Therefore μ_u is right-continuous in t.

5. *In general, the distribution function μ_u is not left- continuous. Indeed*

$$\lim_{h\to 0^+}(\mu_u(t-h)-\mu_u(t)) = |\{u=t\}|$$

and so μ_u is left-continuous at t if and only if $|\{u=t\}|=0$.

Theorem 1.4.3. *Let u be a measurable non-negative function defined in Ω which vanishes on $\partial\Omega$. Let $F:\mathbb{R}^+\to\mathbb{R}$ be a non-negative increasing differentiable function such that $F(0)=0$. We have that*

$$\int_\Omega F(u(x))\ dx = \int_0^\infty f(s)|\{u\geq s\}|\ ds. \tag{1.4.2}$$

where f is such that $f=F'$ in $\mathbb{R}^+$.

Proof: We have

$$\begin{aligned}\int_\Omega F(u(x))\ dx &= \int_\Omega\left(\int_0^{u(x)} f(s)\ ds\right)\ dx\\ &= \int_0^\infty f(s)\left(\int_{\{u\geq s\}} dx\right)\ ds \qquad \text{by Fubini's theorem}\\ &= \int_0^\infty f(s)|\{u\geq s\}|\ ds.\end{aligned}$$

□

Consider $\Omega\subset\mathbb{R}^n$ an open set and let $E\subset\Omega$ be a measurable set. Denote by $P_\Omega(E)$ the area of the boundary surface of E contained in Ω.

Theorem 1.4.4. *(Fleming Rischell) Let $\Omega\subset\mathbb{R}^n$ be an open set and let $u\in W^{1,1}(\Omega)$. Then*

$$\int_\Omega |\nabla u| dx = \int_{-\infty}^{+\infty} P_\Omega(\{u>t\})dt.$$

Proof: See Kesavan [17]. □

More generally, we have

Theorem 1.4.5. *Let u be a function in $W^{1,p}(\mathbb{R}^n)$ and $g : \mathbb{R}^n \to \mathbb{R}$ a non-negative Borel function. Then*

$$\int_{\mathbb{R}^n} g(x)|\nabla u(x)|\ dx = \int_{-\infty}^{+\infty} \left(\int_{\{u=s\}} g(x)\ d\sigma \right) ds. \tag{1.4.3}$$

Proof: See [5]. □

Definition 1.4.6. *Let $A \subset \mathbb{R}^n$ be measurable. The **Lebesgue density** of A in $x \in \mathbb{R}^n$ is defined as*

$$D(A, x) = \lim_{r \to 0} \frac{|A \cap B(x, r)|}{|B(x, r)|}.$$

where $B(x, r)$ denotes the closed ball of radius r centered in x.

We remark if $x \in Int(A)$ then $D(A, x) = 1$, and if $x \in Int(A^c)$ then $D(A, x) = 0$. Consider $E \subset \mathbb{R}^n$ measurable. The **measure theoretic boundary** of E is given by

$$\partial^* E = \{x : 0 < D(E, x) < 1\}.$$

Thus, we say that E has **finite perimeter** if $\mathrm{H}^{n-1}(\partial^* E) < \infty$, where H^{n-1} is the $n-1$ dimensional Hausdorff measure .

1.5 Other results

The next proposition is a topological result.

Proposition 1.5.1. *Let A, B be nonempty open sets in $\mathbb{R}^n$, such that B is connected. We have*

$$A \subset B, \partial A \subset \partial B \Rightarrow A = B.$$

Proof: As A is an open set in $\mathbb{R}^n$ and $A = A \cap B$, we get that A is an open set for the relative topology on B.

Let $\partial_B A$ the boundary of A with respect to the relative topology in B. We first prove that $\partial_B A \subset \partial A \cap B$. Let $x \in \partial_B A$. Then, $x \in B$. Let V any open neighborhood of x. Then $V_B = V \cap B$ is a neighborhood of x in B and since $x \in \partial_B A$, have

$$V_B \cap A \neq \emptyset \text{ and } V_B \cap A^c \neq \emptyset.$$

Therefore,

$$V \cap A \neq \emptyset \text{ and } V \cap A^c \neq \emptyset.$$

This shows that $x \in \partial A$ and $x \in B$, and thus $\partial_B A \subset \partial A \cap B$.

Now we show that $\partial A \cap B \subset \partial_B A$. Let $x \in \partial A \cap B$ and $V_B = V \cap B$ be a neighborhood of $x \in B$ where V is an open set in $\mathbb{R}^n$. Note that as V, B are open sets, $V \cap B$ is also open in $\mathbb{R}^n$. Since, $x \in \partial A \cap B$, we have

$$V_B \cap A \neq \emptyset \wedge V_B \cap A^c \neq \emptyset.$$

Thus $x \in \partial_B A$. We deduce that

$$\partial_B A = \partial A \cap B. \tag{1.5.1}$$

Now note that as B is an open set in $\mathbb{R}^n$, $B \cap \partial B = \emptyset$ and as by hypothesis $\partial A \subset \partial B$, we have $\partial A \cap B = \emptyset$. By (1.5.1), $\partial_B A = \emptyset$. Thus, if $\overline{A}^B$ is the closure of A relative to B, then

$$\overline{A}^B = A \cup \partial_B A = A.$$

In this way, A is relatively closed in B. But, we saw that A is relatively open in B, and as B is connected, we conclude that $A = B$ or $A = \emptyset$. Therefore $A = B$. □

Proposition 1.5.2. *Let $\omega_1, \omega_2 \in \mathbb{R}^n$. If $p > 1$, then*

$$|\omega_2|^p \geq |\omega_1|^p + p|\omega_1|^{p-2}\omega_1 \cdot (\omega_2 - \omega_1). \tag{1.5.2}$$

with equality if and only if $\omega_1 = \omega_2$. This is just the strict convexity of the function $\omega \to |\omega|^p$.

Proposition 1.5.3. *Let* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be given by*

$$A(x) = |x|^{p-2}x.$$

Then the matrix $\mathsf{A} = \left(\frac{\partial A_i}{\partial x_j}(x) \right)_{i,j=1}^{n}$ *satsifies the inequality*

$$\langle \mathsf{A}\xi, \xi \rangle \geq min\{1, p-1\}|x|^{p-2}|\xi|^2.$$

Proof: See Chorwadwala et. al. [9]. □

Chapter 2

Tools

2.1 Schwarz symmetrization

This section is based on Kawohl [16] and Kesavan [18]. We define the Schwarz symmetrization of a domain Ω and of a non-negative measurable function u defined in Ω.

Definition 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable non-empty set. The* ***Schwarz symmetrization*** *of Ω denoted by Ω^* is defined as the open ball centered at the origin and having the same Lebesgue measure as Ω. If $\Omega = \emptyset$, define its Schwarz symmetrization as $\Omega^* = \emptyset$.*

The following proposition is obvious from the definition of Schwarz symmetrization of a domain.

Proposition 2.1.2. *Let $A, B \subset \mathbb{R}^n$ be a measurable non-empty sets. If $A \subset B$, then $A^* \subset B^*$.*

Now we define the Schwarz symmetrization of certain kinds of functions.

Definition 2.1.3. *Let $\Omega \subset \mathbb{R}^n$ be a measurable nonempty set and let $u : \Omega \to \mathbb{R}$ be a non-negative bounded measurable function, which vanishes on $\partial\Omega$. The* ***Schwarz***

symmetrization *of u denoted by u^* is the function defined in Ω^* as*

$$u^*(x) = \sup\{c : x \in \Omega_c^*\}.$$

where Ω_c is the level set of u defined in (1.4.1).

Remark 2.1.4. *Let $x \in \Omega^*$. Observe that $u^*(x)$ is well defined for the following reasons:*

1. *As $u \geq 0$, for any $c \leq 0$ we have $\Omega_c^* = \Omega^*$. It follows that $\{c : x \in \Omega_c^*\} \neq \emptyset$.*
2. *For any $c \geq \sup u$ we have $\Omega_c = \emptyset$. Thus, $\Omega_c^* = \emptyset$ and $\{c : x \in \Omega_c^*\}$ is upper bounded by $\sup u$.*

Therefore, u^* is well defined and

$$0 \leq u^* \leq \sup u. \tag{2.1.1}$$

Proposition 2.1.5.

$$\sup_{\Omega^*} u^* = \sup_{\Omega} u \tag{2.1.2}$$

Proof: By (2.1.1)

$$u^*(x) \leq \sup_{\Omega} u.$$

Further, $u^*(0) = \sup_{\Omega} u$, since $0 \in \Omega_c^*$, for all $c < \sup_{\Omega} u$. Therefore (2.1.2) holds. □

Proposition 2.1.6. *The function u^* is radially symmetric and non-increasing with respect to the radius.*

Proof: Let $x, y \in \Omega^*$. Suppose $|x| = |y|$. If $x \in \Omega_c^*$ for some c, $|y| = |x| < R_c$ where R_c is the radius of Ω_c^*. So, $y \in \Omega_c^*$ and viceversa. So $\{c : x \in \Omega_c^*\} = \{c : y \in \Omega_c^*\}$ and it follows that $u^*(x) = u^*(y)$, showing that u^* is radially symmetric.

If $|y| \leq |x|$, in a similar way to the first case we have $\{c : x \in \Omega_c^*\} \subset \{c : y \in \Omega_c^*\}$, and this implies that $u^*(x) \leq u^*(y)$, showing that u^* is radially nonincreasing. □

Note that since u^* is radially symmetric, the level sets $\{u^* > c\}$ are balls.

Proposition 2.1.7. *Let $c \in \mathbb{R}$. Then*

$$\{u^*(x) \geq c\} = \bigcap_{b<c} \Omega_b^*. \tag{2.1.3}$$

.

Proof: $(\Rightarrow)$

Let $x \in \{u^*(x) \geq c\}$. Let b be fixed such that $b < c$. We deduce that $u^*(x) > b$ and so, there exists $b < s$ for which $x \in \{u \geq s\}^* \subset \{u \geq b\}^*$. As $b < c$ is arbitrary, then $x \in \bigcap_{b<c} \Omega_b^*$. $(\Leftarrow)$

Let $x \in \bigcap_{b<c} \Omega_b^*$. We get $\{b : b < c\} \subset \{s : x \in \Omega_s^*\}$. Thus

$$c = \sup\{b : b < c\} \leq \sup\{s : x \in \Omega_s^*\} = u^*(x).$$

proving that $x \in \{u^*(x) \geq c\}$. □

Proposition 2.1.8. *Let $c \in \mathbb{R}$. Given a non-negative measurable function u on Ω, we have*

$$|\{u^* \geq c\}| = |\{u \geq c\}|. \tag{2.1.4}$$

That is, the level sets of u^* and u are equimeasurable.

Proof: We consider 3 cases for fixed c:

1. $c \leq 0$.

   Since $u \geq 0$, so $u^* \geq 0$. Then

   $$\{u \geq c\} = \Omega \text{ and } \{u^* \geq c\} = \Omega^*. \tag{2.1.5}$$

   Therefore

   $$|\{u^* \geq c\}| = |\Omega^*| = |\Omega| = |\{u \geq c\}|$$

   .

2. $c > \sup u$

   In this case
$$\{u \geq c\} = \emptyset, \tag{2.1.6}$$
   and by (2.1.1), $u^*(x) \leq \sup_\Omega u < c$, which implies that
$$\{u^* \geq c\} = \emptyset. \tag{2.1.7}$$
   Thus,
$$|\{u \geq c\}| = 0 = |\{u^* \geq c\}|.$$

3. $0 < c \leq supu$

   We first note that
$$\{u \geq c\} = \bigcap_{n=1}^{\infty} \left\{u \geq c - \frac{1}{n}\right\}, \tag{2.1.8}$$
   and the sequences $\left\{u \geq c - \frac{1}{n}\right\}_{n=1}^{\infty}$ and $\left\{u \geq c - \frac{1}{n}\right\}_{n=1}^{*\infty}$ are decreasing. Thus, by (2.1.3),
$$\begin{aligned} |\{u^* \geq c\}| &= \left|\bigcap_{n=1}^{\infty} \left\{u \geq c - \frac{1}{n}\right\}^*\right| \\ &= \lim_{n\to\infty} \left|\left\{u \geq c - \frac{1}{n}\right\}^*\right| \\ &= \lim_{n\to\infty} \left|\left\{u \geq c - \frac{1}{n}\right\}\right| \\ &= |\{u \geq c\}| . \end{aligned}$$

□

Remark 2.1.9. *We note that as*
$$\{u > s\} = \bigcup_{n=1}^{\infty} \left\{u \geq s + \frac{1}{n}\right\}$$
and
$$\{u^* > s\} = \bigcup_{n=1}^{\infty} \left\{u^* \geq s + \frac{1}{n}\right\}$$

then by Proposition 2.1.8,

$$|\{u > s\}| = \lim_{n\to\infty} \left|\left\{u \geq s + \frac{1}{n}\right\}\right| = \lim_{n\to\infty} \left|\left\{u^* \geq s + \frac{1}{n}\right\}\right| = |\{u^* > s\}|.$$

Finally, we may also conclude that

$$|\{u = s\}| = |\{u^* = s\}|.$$

From Proposition 2.1.8, the next proposition is deduced.

Proposition 2.1.10. *Let u be a measurable bounded non-negative function defined in Ω which vanishes on $\partial\Omega$. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a non-negative Borel function such that $F(0) = 0$. We have then*

$$\int_\Omega F(u(x))\ dx = \int_{\Omega^*} F(u^*(x))\ dx. \tag{2.1.9}$$

Proof: We give the proof when F is differentiable. Let $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f' = F$ in $\mathbb{R}^+$. By Theorem (1.4.3) and since u, u^* are equimeasurable we obtain

$$\int_\Omega F(u(x))\ dx = \int_0^\infty f(s)|\{u \geq s\}|\ ds = \int_0^\infty f(s)|\{u^* \geq s\}|\ ds = \int_{\Omega^*} F(u^*(x))\ dx.$$

For the general case, see [17]. □

Proposition 2.1.11. *Let $F : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a convex lower semicontinuous function with $F(0) = 0$. Let u, v be measurable bounded non-negative functions defined in a bounded domain Ω which vanishes in $\partial\Omega$. Let u^*, v^* be their respective Schwarz symmetrizations. Then,*

$$\int_{\Omega^*} F(|u^* - v^*|) \leq \int_\Omega F(|u - v|)$$

Proof: See Chiti [8]. □

The proofs of the following two theorems are as in Kesavan [?].

Theorem 2.1.12. *Let $\Omega \subset \mathbb{R}^n$ be a measurable open set and let $u \in D(\Omega)$. If $u \geq 0$, then for $1 \leq p < \infty$, we have*

$$\int_{\Omega} |\nabla u|^p dx = \int_0^M \left(\int_{\{u=t\}} |\nabla u|^{p-1} \, d\sigma \right) \, dt \tag{2.1.10}$$

where $M = \max_{x \in \overline{\Omega}} u(x)$. Also, if $u^* : \Omega^* \to \mathbb{R}$ is the Schwarz symmetrization, then

$$\int_{\Omega^*} |\nabla u^*|^p dx = \int_0^M \left(\int_{\{u^*=t\}} |\nabla u^*|^{p-1} \, d\sigma \right) \, dt \tag{2.1.11}$$

Proof: Step 1. Since u is smooth, by Sard's theorem, for almost every t in the range of u, we have $|\nabla u| \neq 0$ on $\{u = t\}$. Thus, $\{u = t\}$ will be an $(n-1)$- dimensional surface and, further $\{u = t\} = \partial\{u > t\}$ and $|\{u^* = t\}| = |\{u = t\}| = 0$.

Step 2. Let $2 \leq p < \infty$. We define

$$f = -\text{div} \left(|\nabla u|^{p-2} \nabla u \right).$$

Then, for all $v \in W_0^{1,p}(\Omega)$, we have

$$\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla v \rangle \, dx = \int_{\Omega} f v \, dx. \tag{2.1.12}$$

Let $t > 0$. Set $v = (u - t)^+ \in W_0^{1,p}(\Omega)$. We observe that $v \neq 0$ in $\{u > t\}$ and that on this set $v = u - t$. Using v as test function in (2.1.12), we obtain

$$\int_{\{u>t\}} |\nabla u|^p \, dx = \int_{\{u>t\}} f(u - t) \, dx. \tag{2.1.13}$$

Thus, differentiating with respect to t and using Lemma 2.2.1 in [17], we get

$$\frac{d}{dt} \left(\int_{\{u>t\}} |\nabla u|^p \, dx \right) = - \int_{\{u>t\}} f \, dx. \tag{2.1.14}$$

By (2.1.14)

$$\begin{aligned}\int_{\Omega} |\nabla u|^p \, dx &= \int_{\{M \geq u > 0\}} |\nabla u|^p \, dx + \int_{\{u=0\}} |\nabla u|^p \, dx \\ &= \int_{\{M \geq u > 0\}} |\nabla u|^p \, dx \\ &= -\left(\int_{\{u > M\}} |\nabla u|^p \, dx - \int_{\{u>0\}} |\nabla u|^p \, dx \right) \\ &= -\int_0^M \frac{d}{dt} \left(\int_{\{u>t\}} |\nabla u|^p \, dx \right) \, dt \\ &= \int_0^M \left(\int_{\{u>t\}} f \, dx \right) \, dt. \end{aligned} \tag{2.1.15}$$

Let t be such that $\nabla u \neq 0$ on $\{u = t\}$. We note that on $\{u = t\}$

$$\nabla u = \frac{\partial u}{\partial n} n + \nabla_{\{u=t\}} u,$$

where $\nabla_{\{u=t\}} u$ is the tangential gradient. Since u is constant on $\{u = t\}$, we have $\nabla_{\{u=t\}} u = 0$. Thus

$$\nabla u = \frac{\partial u}{\partial n} n.$$

on $\{u = t\}$.

As $u > t$ inside of $\{u > t\}$ and $u = t$ on $\partial\{u > t\}$, the normal derivative $\frac{\partial u}{\partial n} \leq 0$ in this set. Thus, on $\{u = t\}$,

$$|\nabla u| = -\frac{\partial u}{\partial n} = -\langle \nabla u, n \rangle. \tag{2.1.16}$$

By this, the definition of f, and integration by parts we get

$$\int_{\{u>t\}} f \, dx = -\int_{\{u=t\}} |\nabla u|^{p-2} \langle \nabla u, n \rangle \, d\sigma = \int_{\{u=t\}} |\nabla u|^{p-1} \, d\sigma. \tag{2.1.17}$$

Plugging this in (2.1.15), we get (2.1.10) for $2 \leq p < \infty$.

Step 3. Let $1 \leq p < 2$. We wish to imitate the previous method of proof but $|\nabla u|^{p-2}$ becomes infinite if the gradient vanishes. So we use an approximation technique. Let $\varepsilon > 0$. Define

$$f_\varepsilon = -\mathrm{div} \left(\left(|\nabla u|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla u \right).$$

Thus, for all $v \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega \left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} \langle \nabla u, \nabla v\rangle \, dx = \int_\Omega f_\varepsilon v \, dx. \tag{2.1.18}$$

Choosing $v = (u-t)^+$ for $t > 0$ as in Step 2 we obtain

$$\int_\Omega \left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u|^2 \, dx = \int_0^M \left(\int_{\{u=t\}} \left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u| \, d\sigma \right) dt. \tag{2.1.19}$$

Since $1 \le p < 2$,

$$\left(\frac{|\nabla u|^2}{|\nabla u|^2 + \varepsilon} \right)^{\frac{2-p}{2}} < 1,$$

thus

$$\left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u|^2 = \left(\frac{|\nabla u|^2}{|\nabla u|^2 + \varepsilon} \right)^{\frac{2-p}{2}} |\nabla u|^p \le |\nabla u|^p, \tag{2.1.20}$$

By this, we can apply the dominated convergence theorem on the left-hand side of (2.1.19), obtaining

$$\lim_{\varepsilon \to 0} \int_\Omega \left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u|^2 \, dx = \int_\Omega |\nabla u|^p. \tag{2.1.21}$$

Similarly, we have

$$\left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u| \le |\nabla u|^{p-1},$$

which is integrable on the set $\{u = t\}$ for almost every t. Now, by the dominated convergence theorem

$$\lim_{\varepsilon \to 0} \int_{\{u=t\}} \left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u| \, d\sigma = \int_{\{u=t\}} |\nabla u|^{p-1} \, d\sigma.$$

Moreover,

$$\int_{\{u=t\}} \left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u| \, d\sigma \le \int_{\{u=t\}} |\nabla u|^{p-1} \, d\sigma$$

and

$$\int_0^M \left(\int_{\{u=t\}} |\nabla u|^{p-1} \, d\sigma \right) dt < \infty$$

since $u \in D(\Omega)$. Thus, applying the dominated convergence theorem on the right-hand side of (2.1.19), we have

$$\lim_{\varepsilon \to 0} \int_0^M \left(\int_{\{u=t\}} \left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u| \, d\sigma \right) dt = \lim_{\varepsilon \to 0} \int_0^M \left(\int_{\{u=t\}} |\nabla u|^{p-1} \, d\sigma \right) dt. \tag{2.1.22}$$

Therefore letting tends $\varepsilon \to 0$ in (2.1.19), from (2.1.21) and (2.1.22), we obtain (2.1.10) for $1 \le p < 2$.

Step 4. Let R be the radius of Ω^*. We write $u^*(x) = u^*(|x|)$ and in this way, we can consider u^* as a function of a single variable r. Since it is a nonincreasing function, u^* is differentiable almost everywhere in $[0, R]$. Using polar coordinates, we obtain

$$\begin{aligned}\int_{\Omega^*} |\nabla u^*|^p &= \int_0^R |u^{*\prime}(r)|^p n\omega_n r^{n-1}\, dr \\ &= \int_0^R |u^{*\prime}(r)|^{p-1} n\omega_n r^{n-1}(-u^{*\prime}(r))\, dr \\ &= \int_0^M |\nabla u^*_{|\{u^*=t\}|}|^{p-1} |\{u^* = t\}|_{n-1}\, dt \\ &= \int_0^M \left(\int_{\{u^*=t\}} |\nabla u^*|^{p-1}\, d\sigma \right) dt.\end{aligned}$$

using the change of variables $t = u^*(r)$, and observing that $|\{u^* = t\}|_{n-1} = n\omega_n (r(t))^{n-1}$ and the gradient of u^* is constant on each of the sets $\{u^* = t\}$. □

Theorem 2.1.13. *Let $u \in D(\Omega)$ be such that $u \ge 0$. Let μ_u be the distribution function of u. We have that*

$$-\mu_u'(t) = \int_{\{u=t\}} \frac{d\sigma}{|\nabla u|} = \int_{\{u^*=t\}} \frac{d\sigma}{|\nabla u^*|} \tag{2.1.23}$$

for almost every t in the range of u.

Proof: Step 1: Since u is smooth, by Sard's theorem, for almost every t in the range of u, we have $|\nabla u| \ne 0$ on $\{u = t\}$. Thus, $\{u = t\}$ will be an $(n-1)$- dimensional surface and, further $\{u = t\} = \partial\{u > t\}$ and $|\{u^* = t\}| = |\{u = t\}| = 0$.

Step 2. Let $\varepsilon > 0$. Define

$$f_\varepsilon = -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|^2 + \varepsilon} \right).$$

Let t such that $\nabla u \ne 0$ on $\{u = t\}$ as Step 1. As $\partial\{u > t\} = \{u = t\}$, we have $(u - t)^+ = 0$ in $\partial\{u > t\}$. Thus, multiplying f by $(u - t)^+$ and applying integration by parts, we get

$$\int_{\{u>t\}} \frac{\nabla u}{|\nabla u|^2 + \varepsilon}\, dx = \int_{\{u>t\}} f_\varepsilon (u - t)\, dx.$$

Differentiating with respect to t and using Lemma 2.2.1 in [17], we obtain that

$$-\frac{d}{dt}\int_{\{u>t\}} \frac{|\nabla u|^2}{|\nabla u|^2+\varepsilon}\, dx = \int_{\{u>t\}} f_\varepsilon\, dx.$$

Step 3. Consider a t with same characteristics as in Step 2, i.e., $|\nabla u| \neq 0$ on the set $\{u = t\}$. For sufficiently small $h > 0$, the same holds for the set $\{t - h \leq u \leq t + h\}$. Thus

$$\int_{\{t-h<u\leq t\}} \frac{|\nabla u|^2}{|\nabla u|^2+\varepsilon}\, dx = \int_{t-h}^{t} \left(-\frac{d}{dt}\int_{\{u>\tau\}} \frac{|\nabla u|^2}{|\nabla u|^2+\varepsilon}\, dx\right) d\tau = \int_{t-h}^{t} \left(\int_{\{u>\tau\}} f_\varepsilon\, dx\right) d\tau. \tag{2.1.24}$$

In a similar way as was proved in the aseveration (2.1.17) in the Theorem 2.1.12, we obtain

$$\int_{\{u>\tau\}} f_\varepsilon\, dx = \int_{\{u=\tau\}} \frac{|\nabla u|}{|\nabla u|^2+\varepsilon}\, d\sigma.$$

By this and by (2.1.24),

$$\int_{\{t-h<u\leq t\}} \frac{|\nabla u|^2}{|\nabla u|^2+\varepsilon}\, dx = \int_{t-h}^{t} \left(\int_{\{u>\tau\}} f_\varepsilon\, dx\right) d\tau = \int_{t-h}^{t} \left(\int_{\{u=\tau\}} \frac{|\nabla u|}{|\nabla u|^2+\varepsilon}\, d\sigma\right) d\tau. \tag{2.1.25}$$

Applying the dominated convergence theorem as ε tends to zero, it follows that

$$\mu_u(t-h) - \mu_u(t) = \int_{t-h}^{t} \left(\int_{\{u=\tau\}} \frac{d\sigma}{|\nabla u|}\right) d\tau. \tag{2.1.26}$$

Analogously we can get

$$\mu_u(t) - \mu_u(t+h) = \int_{t}^{t+h} \left(\int_{\{u=\tau\}} \frac{d\sigma}{|\nabla u|}\right) d\tau. \tag{2.1.27}$$

Finally, dividing by h and taking the limit as $h \to 0$ in (2.1.26) and (2.1.27), we obtain the first relation in (2.1.23).

Step 4. Let $r(t)$ be the radius of the ball $\{u^* > t\}$. Since $\mu_u(t)$ and $(r(t))$ are nonincreasing functions, they are differentiable for almost every t. Moreover, as u, u^* are equimeasurables functions $\mu_u(t) = \omega_n (r(t))^n$ and so,

$$\mu_u'(t) = n\omega_n (r(t))^{n-1} r'(t). \tag{2.1.28}$$

By abuse of notation, we can write $u^*(x) = u^*(|x|)$, thus we can say that $u^*(r(t)) = t$ for almost every t. Applying implicit differentiation we obtain

$$r'(t) = \frac{1}{u^{*\prime}(r(t))}.$$

Replacing this in (2.1.28) and considering the fact that $|\{u^* = t\}|_{n-1} = n\omega_n(r(t))^{n-1}$ we have

$$\mu_u'(t) = |\{u^* = t\}|_{n-1}\frac{1}{u^{*\prime}(r(t))}. \tag{2.1.29}$$

Since u^* is decreasing and radially symmetric, we have $u^{*\prime}(r(t)) = -|\nabla u^*_{|\{u^*=t\}}|$. So,

$$\mu_u'(t) = -\frac{|\{u^* = t\}|_{n-1}}{|\nabla u^*_{|\{u^*=t\}}|} = \int_{\{u^*=t\}} \frac{d\sigma}{|\nabla u^*|}.$$

obtaining the second part of (2.1.23). □

Now we can prove one of the most important properties of Schwarz symmetrization.

Theorem 2.1.14. *(PÃşlya - SzegÃű) Let $1 \le p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u \in W_0^{1,p}(\Omega)$ such that $u \ge 0$. Then*

$$\int_{\Omega^*} |\nabla u^*|^p dx \le \int_\Omega |\nabla u|^p dx. \tag{2.1.30}$$

In particular, $u^* \in W_0^{1,p}(\Omega^*)$.

Proof: 1. The case $p = 1$.

Since $u \ge 0$ in Ω and $u = 0$ on $\partial\Omega$, we have $P_\Omega(\{u > t\}) = P_{\mathbb{R}^n}(\{u > t\})$ for $t > 0$. Moreover, since $\{u > t\}$ and $\{u^* > t\}$ have the same measure, by the classical isoperimetric inequality, we have

$$P_{\mathbb{R}^n}(\{u > t\}) \ge P_{\mathbb{R}^n}(\{u^* > t\}).$$

Thus by Theorem 1.4.4 and by the fact $u^* \in W^{1,1}(\Omega)$(see ([17])), it follows that

$$\int_\Omega |\nabla u| dx = \int_0^{+\infty} P_{\mathbb{R}^n}(\{u > t\}) dt \ge \int_0^{+\infty} P_{\mathbb{R}^n}(\{u^* > t\}) dt = \int_{\Omega^*} |\nabla u^*| dx.$$

2. The case $1 < p < \infty$

Let $u \in D(\Omega)$ such that $u \geq 0$. Let $M = \max_{x\in\overline{\Omega}} u(x)$. Following Theorem (2.1.12), it is enough to prove that

$$\int_{\{u^*=t\}} |\nabla u^*|^{p-1}\, d\sigma \leq \int_{\{u=t\}} |\nabla u|^{p-1}\, d\sigma. \tag{2.1.31}$$

Now we show (2.1.31). Since u is smooth, we can assume by the Sard's theorem that $|\nabla u|$ does not vanish in $\{u = t\}$ for almost every $t \in (0, M)$. Define a measure v on $\{u = t\}$ as $dv = \frac{d\sigma}{|\nabla u|}$. By Jensen's inequality, we get

$$\begin{aligned} \int_{\{u=t\}} |\nabla u|^{p-1}\, d\sigma &= \int_{\{u=t\}} |\nabla u|^{p-1}\, dv \\ &\geq \frac{\left(\int_{\{u=t\}} |\nabla u|\, dv\right)^p}{\left(\int_{\{u=t\}} dv\right)^{p-1}} \\ &= \frac{\left(\int_{\{u=t\}} d\sigma\right)^p}{\left(\int_{\{u=t\}} dv\right)^{p-1}} \\ &= \frac{(|u=t|)_{n-1}^p}{\left(\int_{\{u=t\}} dv\right)^{p-1}} \end{aligned} \tag{2.1.32}$$

Moreover, by the classical isoperimetric inequality,

$$|\{u = t\}|_{n-1} \geq |\{u^* = t\}|_{n-1} \tag{2.1.33}$$

where $|\{u = t\}|_{n-1}$ and $|\{u^* = t\}|_{n-1}$ are the $n-1$ dimensional Hausdorff measures on $\{u = t\}$ and $\{u^* = t\}$, respectively. As u^* is radially symmetric, we have ∇u^* is constant in $\{u^* = t\}$. Combining this with (2.1.32),(2.1.33) and Theorem 2.1.13

we obtain

$$\begin{aligned}\int_{\{u=t\}} |\nabla u|^{p-1}\, d\sigma &\geq \frac{(|u^*=t|)_{n-1}^p}{(-\mu'(t))^{p-1}} \\ &\geq \frac{(|u^*=t|)_{n-1}^p}{\left(\int_{\{u^*=t\}} \frac{d\sigma}{|\nabla u^*|}\right)^{p-1}} \\ &= |\{u^*=t\}|_{n-1} |\nabla u^*_{|_{\{u^*=t\}}}|^{p-1} \\ &= \int_{\{u^*=t\}} |\nabla u^*|^{p-1}\, d\sigma.\end{aligned}$$

Therefore we have proved (2.1.30) for $u \in D(\Omega)$.

Now let $u \in W_0^{1,p}(\Omega)$ be such that $u \geq 0$. By the definition of $W_0^{1,p}(\Omega)$, there exists $u_n \in D(\Omega), u_n \geq 0$ and $u_n \to u$ in $W_0^{1,p}(\Omega)$. We have

$$\int_{\Omega^*} |\nabla u_n^*|^p dx \leq \int_\Omega |\nabla u_n|^p dx. \tag{2.1.34}$$

The sequence $\{u_n^*\}$ is bounded in $W_0^{1,p}(\Omega^*)$ and we can assume that is weakly convergent to a function v in $W^{1,p}(\Omega^*)$ and hence, by Rellich theorem, $u_n^* \to v$ in $L^p(\Omega^*)$. On the other hand, since $u_n \to u$ in $L^p(\Omega)$, applying Proposition 2.1.11 with $F(s) = s^p$, we have $u_n^* \to u^*$ in $L^p(\Omega^*)$. Therefore $u^* = v$. We pass to the limit in (**??**), using the lower semicontinuity of the $W_0^{1,p}$ norm for the weak convergence of u_n^* to u and the fact that u_n converges strongly to u in $W_0^{1,p}(\Omega)$, to get

$$\int_{\Omega^*} |\nabla u^*|^p dx \leq \liminf \int_{\Omega^*} |\nabla u_n^*|^p dx \leq \liminf \int_\Omega |\nabla u_n|^p dx = \int_\Omega |\nabla u|^p dx.$$

□

Now we shall examine under what conditions the equality holds in the Polya Szego inequality, i.e. , what conditions must satisfy a domain Ω and a function u defined in Ω, such that the equality holds in (2.1.30).

Consider $1 \leq p < \infty$ and let $A : [0, \infty) \to [0, \infty)$ be a function belonging to $C^2([0, \infty))$ such that $A^{\frac{1}{p}}$ is convex and $A(0) = 0$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable bounded non-negative function with compact support. Assume that ∇u in the weak sense is a measurable function with

$$\int_{\mathbb{R}^n} A(|\nabla u|) < \infty. \tag{2.1.35}$$

Denote $M = \sup u = \sup u^*$ and $C^* = \{x \in \Omega^* : \nabla u^*(x) = 0\}$. Now, we enunciate the following theorem of general character.

Theorem 2.1.15. *If u satisfies (2.1.35), then ∇u^* is a measurable function and*

$$\int_{\mathbb{R}^n} A(|\nabla u^*|) \leq \int_{\mathbb{R}^n} A(|\nabla u|). \tag{2.1.36}$$

Moreover, if $1 < p < \infty, |C^* \cap {u^*}^{-1}(0, M)| = 0,$ A is strictly increasing, and the equality holds in (2.1.36), then there exists a translate of u^* which is equal almost everywhere to u.

Proof: See Brothers and Ziemer [5]. □

We may take $A(\xi) = |\xi|^p$, which is strictly convex, strictly increasing and such that $A(0) = 0$. Thus, we obtain the following corollary.

Corollary 2.1.16. *Let $1 < p < \infty$. Let Ω be a bounded domain in $\mathbb{R}^n$. Let u be a bounded non negative function in $W_0^{1,p}(\Omega)$. If $|C^* \cap {u^*}^{-1}(0, M)| = 0$ and*

$$\int_{\Omega^*} |\nabla u^*|^p dx = \int_{\Omega} |\nabla u|^p dx. \tag{2.1.37}$$

Then there exists a translate of u^* which is equal almost everywhere to u.

By the above corollary we also have that Ω is a ball congruent to Ω^*.

2.2 Steiner symmetrization

This section is based on Kawohl [16].

Definition 2.2.1. *Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a measurable non-empty set. Denote by Ω' the orthogonal projection of Ω in $\mathbb{R}^{n-1}$,i.e.:*

$$\Omega' := \{x' \in \mathbb{R}^{n-1} : \exists y \text{ such that } (x', y) \in \Omega\},$$

and for $x' \in \mathbb{R}^{n-1}$, denote by $\Omega(x')$ the intersection of Ω with $\{x'\} \times \mathbb{R}$:

$$\Omega(x') := \{y \in \mathbb{R} : (x', y) \in \Omega\}.$$

Define the ***Steiner symmetrization*** *Ω^s of Ω with respect to hyperplane $y = 0$ as the set*

$$\Omega^s = \bigcup_{x' \in \Omega'} \Omega^s(x')$$

where

$$\Omega^s(x') = \left\{(x', y) \in \mathbb{R}^n : |y| < \frac{1}{2} |\Omega(x')| \right\}.$$

If $\Omega = \emptyset$, define its ***Steiner symmetrization*** *by $\Omega^s = \emptyset$.*

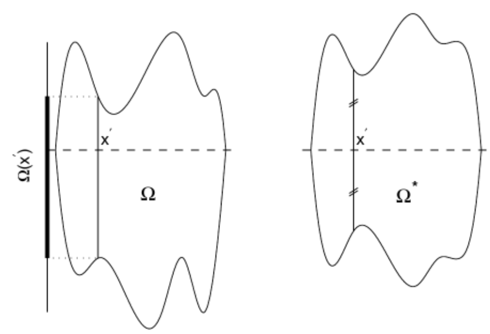


Figure 2.1: On the left hand, a domain Ω; on the right hand, its Steiner symmetrization Ω^s with respect the hyperplane $y = 0$

In simple words, Ω^s is obtained from Ω by putting the midpoint of each segment $\Omega(x')$ at $(x', 0)$. Clearly, Ω^s is symmetric with respect to the hyperplane $y = 0$.

Proposition 2.2.2. *Let $A, B \subset \mathbb{R}^n$ be a measurable non-empty sets. If $A \subset B$ then $A^s \subset B^s$.*

Proof: Let $(x', y) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$. If $A \subset B$ then $A(x') \subset B(x')$, and so $|A(x')| \leq |B(x')|$. Thus

$$(x', y) \in A^s \Rightarrow |y| < \frac{1}{2}|A(x')| \leq \frac{1}{2}|B(x')| \Rightarrow (x', y) \in B^s.$$

□

Proposition 2.2.3. *The Steiner symmetrization leaves the volume unchanged, i.e. if Ω is a domain and Ω^s is its Steiner symmetrization, then*

$$|\Omega| = |\Omega^s|$$

Proof: Let Ω' be the orthogonal projection of Ω onto the hyperplane of symmetrization $y = 0$. We note

$$|\Omega| = \int_{\Omega'} |\Omega(x')| \; dx'$$

and

$$|\Omega^s| = \int_{\Omega'} |\Omega^s(x')| \; dx'.$$

As by definition of $\Omega^s(x')$, $|\Omega(x')| = |\Omega^s(x')|$ for all $x' \in \Omega'$, we therefore have

$$|\Omega| = |\Omega^s|.$$

□

Proposition 2.2.4. *The Steiner symmetrization does not increase the perimeter $P(.)$, i.e., if Ω is a domain and Ω^s is its Steiner symmetrization, then*

$$P(\Omega) \geq P(\Omega^s).$$

Proof: For simplicity we prove this result for $\Omega \in \mathbb{R}^2$ being analogous in other dimensions. Consider a given region Ω. We choose the $x - axis$ as the hyperplane of symmetrization. Let Ω' the orthogonal projection of Ω onto $x - axis$. As Ω is a bounded domain , we can assume that Ω' is divided in a finite number $I_1, \ldots, I_l$ of intervals such that for every $x \in I_i$ with $1 \leq i \leq l$, the straight line through $(x, 0)$ which is parallel to the y-axis intersects $\partial\Omega$ in the points $(x, y_1), (x, y_2), \cdots, (x, y_{2m_i})$ where $y_1 > y_2 > \cdots > y_{2m_i}$, with $m_i \geq 1$. The perimeter of Ω is expressed as

$$P(\Omega) = \sum_{i=1}^{l} \int_{I_i} \sum_{v=1}^{2m_i} \sqrt{1 + \left(\frac{\partial y_v}{\partial x}\right)^2}\, dx. \tag{2.2.1}$$

By definition, for every $x \in I_i$ with $1 \leq i \leq l$, we have that the intersection of Ω^s with the straight line through $(x, 0)$ which is parallel to the y-axis corresponds a line-segment bisected by the $x - axis$, with endpoints (x, y_i^s) and $(x, -y_i^s)$, respectively, where

$$y_i^s = \frac{y_1 - y_2 + y_3 - y_4 + \cdots + y_{2m_i - 1} - y_{2m_i}}{2}. \tag{2.2.2}$$

The boundary of Ω^s is bisected by the $x - axis$ and its length $P(\Omega^s)$ can be expressed as

$$\begin{aligned} P(\Omega^s) &= 2\sum_{i=1}^{l} \int_{I_i} \sqrt{1 + \left(\frac{\partial y_i^s}{\partial x}\right)^2}\, dx \\ &= \sum_{i=1}^{l} 2\int_{I_i} \sqrt{1 + \frac{1}{4}\left(\sum_{v=1}^{2m_i} (-1)^{v-1}\frac{\partial y_v}{\partial x}\right)^2}\, dx \\ &= \sum_{i=1}^{l} \int_{I_i} \sqrt{4 + \left(\sum_{v=1}^{2m_i} (-1)^{v-1}\frac{\partial y_v}{\partial x}\right)^2}\, dx. \end{aligned}$$

For every $1 \leq i \leq l$, consider the vectors

$$u_v = \left(1, (-1)^{v-1}\frac{\partial y_v}{\partial x}\right),$$

for $v = 1, 2, \cdots, 2m_i$. The vector $\sum_{v=1}^{2m_i} u_v$ has as first component $2m_i \geq 2$.

So, we have

$$\begin{aligned} \sqrt{4 + \left(\sum_{v=1}^{2m_i} (-1)^{v-1}\frac{\partial y_v}{\partial x}\right)^2} &\leq \left\| \sum_{v=1}^{2m_i} u_v \right\| \\ &\leq \sum_{v=1}^{2m_i} \|u_v\| \\ &= \sum_{v=1}^{2m_i} \sqrt{1 + \left(\frac{\partial y_v}{\partial x}\right)^2}. \end{aligned}$$

Therefore, integrating on every I_i and adding we have

$$P(\Omega) = \sum_{i=1}^{l} \int_{I_i} \sum_{v=1}^{2m_i} \sqrt{1 + \left(\frac{\partial y_v}{\partial x}\right)^2}\, dx \;\; \geq \sum_{i=1}^{l} \int_{I_i} \sqrt{4 + \left(\sum_{v=1}^{2m_i} (-1)^{v-1} \frac{\partial y_v}{\partial x}\right)^2}\, dx = P(\Omega^s).$$

□

Definition 2.2.5. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let $u : \Omega \to \mathbb{R}$ be a bounded measurable non-negative function, which vanishes on $\partial\Omega$. The* ***Steiner symmetrization*** *of u denoted by u^s is a function defined on Ω^s as*

$$u^s(x) = \sup\{c : x \in \Omega_c^s\}.$$

Remark 2.2.6. *We observe that u^s is well defined. The proof of this is the same as in Remark 2.1.4. Also, $u^s \geq 0$ and $\sup_{\Omega^s} u^s = \sup_{\Omega} u$, the latter is proved in the same way as in Proposition 2.1.5.*

Proposition 2.2.7. *The function u^s is symmetric with respect to the hyperplane $y = 0$, i.e. for all $(x', y), (x', y) \in \mathbb{R}^n$,*

$$u^s(x', y) = u^s(x', -y).$$

Proof: Let $(x', y), (x', -y)$ belong to Ω^s. As $|y| = |-y|$, and since Ω_c^s is symmetric with respect to $y = 0$ for every $c \in \mathbb{R}$, we have $(x', y) \in \Omega_c^s$ if and only if $(x', -y) \in \Omega_c^s$. It follows from the definition of u^s that $u^s(x', y) = u^s(x', -y)$. □

The next proposition asserts that u and u^s are equimeasurable functions. Its proof is identical to Proposition 2.1.8 for Schwarz symmetrization , i.e. based on the Proposition (2.1.7). We omit their proofs.

Proposition 2.2.8. *Let $c \in \mathbb{R}$. Given a non-negative measurable function on Ω, we have*

$$|\{u^s \geq c\}| = |\{u \geq c\}|.$$

Remark 2.2.9. *Like in Schwarz symmetrization we have* $|\{u > c\}| = |\{u^s > c\}|$ *and* $|\{u = c\}| = |\{u^s = c\}|$.

The following result is a consequence of the above proposition and its proof is similar to that of Proposition 2.1.10.

Theorem 2.2.10. *Let* $u : \Omega \to \mathbb{R}$ *be a bounded measurable non-negative function which vanishes on* $\partial\Omega$ *and* u^s *its Steiner symmetrization. Let* $F : \mathbb{R}^+ \to \mathbb{R}$ *a measurable function, such that* $F \geq 0$ *and* $F(0) = 0$. *Then*

$$\int_{\Omega} F(u(x))\ dx = \int_{\Omega^s} F(u^s(x))\ dx.$$

Proposition 2.2.11. *Let* $F : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be a convex lower semicontinuous function, with* $F(0) = 0$. *Let* u, v *be bounded measurable non-negative functions defined in a bounded domain* Ω, *which vanishes on* $\partial\Omega$, *and* u^s, v^s *their respective Steiner symmetrizations. Then*

$$\int_{\Omega^s} F(|u^s - v^s|) \leq \int_{\Omega} F(|u - v|)$$

Proof: See Chiti [8]. □

Definition 2.2.12. *Consider*

$$u : \overline{\Omega} \to \mathbb{R}$$
$$(x', y) \mapsto u(x', y)$$

We call that the function u is ***nice*** *if:*

1. $\Omega = \hat{\Omega} \times (-\omega, \omega)$, *where* $\hat{\Omega} \subset \mathbb{R}^{n-1}$ *is a bounded domain.*

2. u *belongs to* $C(\overline{\Omega})$.

3. u is piecewise linear in the sense of affine.

4. $\dfrac{\partial u}{\partial y} \neq 0$ *a.e. in* Ω.

An important property of the Steiner symmetrization is the PÃşlya - SzegÃű inequality, which as we also saw is valid for Schwarz symmetrization. This property is a corollary of the next theorem. We can suppose, without loss of generality, that $\overline{\Omega} = \overline{\Omega^s}$ for a domain Ω.

Theorem 2.2.13. *Let* $\Omega = \Omega' \times (-\omega, \omega)$*, where* $\Omega' \subset \mathbb{R}^{n-1}$ *is a bounded domain. Let* $u : \overline{\Omega} \to \mathbb{R}$ *a non-negative, nice and Lipschitz function in* Ω *such that* $u = 0$ *on* $\partial\Omega$*. Let* $F : \Omega' \times \mathbb{R} \to \mathbb{R}$ *be and* $X_k : \Omega' \to \mathbb{R}$ *with* $k = 1, \cdots n$ *be non-negative and continuous, and let* $G : \mathbb{R}_0^+ \to \mathbb{R}$ *be a nondecreasing and convex function.*

1. *Then*

$$\int_\Omega F(x', u)\; G\left(\left\{\sum_{k=1}^{n-1} X_k(x')\left|\frac{\partial u}{\partial x_k}\right|^2 + X_n(x')\left|\frac{\partial u}{\partial y}\right|^2\right\}^{\frac{1}{2}}\right)\; dx'\; dy$$

$$\geq \int_{\Omega^s} F(x', u^s)\; G\left(\left\{\sum_{k=1}^{n-1} X_k(x')\left|\frac{\partial u^s}{\partial x_k}\right|^2 + X_n(x')\left|\frac{\partial u^s}{\partial y}\right|^2\right\}^{\frac{1}{2}}\right)\; dx'\; dy. \tag{2.2.3}$$

2. *Moreover if* F *and* X_k *with* $k = 1, \cdots, n$ *are positive, and* G *is increasing and strictly convex, then the equality holds in (2.2.3) if and only if* $u = u^s$.

Proof: It is enough to show that for every $x' \in \Omega'$ the following inequality holds

$$\int_{-\omega}^{\omega} I_l\; dy \geq \int_{-\omega}^{\omega} I_r\; dy. \tag{2.2.4}$$

where I_l and I_r are the integrands in the left and right hand sides of (2.2.3). Since u is nice in Ω, we can remove a closed set Γ of $n-1$ dimensional Lebesgue measure zero, such that for every $x' \in \Omega' \setminus \Gamma$, the function $u(x', y)$ is differentiable at all except finitely many $y \in (-w, w)$. Suppose that there exists M such exceptional points and let $a_1 \le a_2 \le \ldots \le a_M$ be the values assumed by u in these M points.

Define

$$D_0 = \{y \in \Omega(x') \mid a_0 = 0 < u(x', y) < a_1\},$$

$$D_i = \{y \in \Omega(x') \mid a_i < u(x', y) < a_{i+1}\} \quad i = 1, \ldots, M-1$$

and

$$D_0^s = \{y \in \Omega(x') \mid a_0 = 0 < u^s(x', y) < a_1\}.$$

Then

$$D_i^s = \{y \in \Omega(x') \mid a_i < u^s(x', y) < a_{i+1}\} \quad i = 1, \ldots, M-1.$$

Fix i and decompose D_i into a finite number of intervals $\{\gamma_{i,j}\}$, where $j = 1, \ldots, N(i, x')$, such that:

1. In each one of the $\gamma_{i,j}$, u is differentiable with respect to all variables and
$$\frac{\partial u}{\partial y}(x', y) \neq 0. \tag{2.2.5}$$
2. $N(i, x')$ must be even, since $u = 0$ on $\partial\Omega$

As $u(x', \cdot)$ is either strictly increasing or decreasing in each $\gamma_{i,j}$, we have for every $\lambda \in (a_i, a_{i+1})$ there exists a unique value of $y_j(\lambda, x')$ in $\gamma_{i,j}$, such that $u(x', y_j) = \lambda$.

On the other hand, as $u^s(x', \cdot)$ is strictly decreasing for $y \in [0, \omega]$, there exists a unique non-negative value $y^s(x', \lambda)$ in D_i^s, such that $u(x', y^s) = \lambda$. Assume that the intervals $\{\gamma_{i,j}\}$ are ordered by their distance from $-\omega$. Thus, as $u(x,' -\omega) = 0$ and $u \ge 0$ is nice, we have u is increasing in $\gamma_{i,1}$, then decreasing $\gamma_{i,2}$ and so on. In this way

$$sign\, \frac{\partial u}{\partial y}(x', y_j) = (-1)^{j+1} sign\, \frac{\partial u}{\partial y}(x', y_1) \quad \text{in} \quad \gamma_{i,j} \tag{2.2.6}$$

Further

$$y^s(x', \lambda) = \frac{1}{2} \sum_{j=1}^{N} (-1)^j y_j(x', \lambda). \tag{2.2.7}$$

On the other hand, we note that the function $y_j(x', \lambda)$ is the inverse function of $u(x', \cdot)$ on $\gamma_{i,j}$, thus $y_j(x', \lambda)$ is differentiable and

$$\frac{\partial u}{\partial y} = \left(\frac{\partial y_j}{\partial \lambda}\right)^{-1}. \tag{2.2.8}$$

Differentiating the relation $u(x', y_j(\lambda, x')) = \lambda$ with respect to x_k with $k = 1, \dots, n-1$, we get by (2.2.8),

$$\frac{\partial u}{\partial x_k} = -\frac{\partial y_j}{\partial x_k}\left(\frac{\partial y_j}{\partial \lambda}\right)^{-1}$$

in $\gamma_{i,j}$. Analogously for u^s and y^s we have that,

$$\frac{\partial u^s}{\partial y} = \left(\frac{\partial y^s}{\partial \lambda}\right)^{-1}$$

and

$$\frac{\partial u^s}{\partial x_k} = -\frac{\partial y^s}{\partial x_k}\left(\frac{\partial y^s}{\partial \lambda}\right)^{-1} \text{ para } k = 1, \dots, n-1,$$

in $D_i^s \cap \{(x', y) | y \geq 0\}$. Moreover, (2.2.6),(2.2.7) and (2.2.8) imply the relations

$$\left|\frac{\partial y^s}{\partial \lambda}\right| = \frac{1}{2}\sum_{j=1}^{N}\left|\frac{\partial y_j}{\partial \lambda}\right| \quad \text{and} \quad \left|\frac{\partial y^s}{\partial x_k}\right| = \frac{1}{2}\left|\sum_{j=1}^{N}(-1)^j\frac{\partial y_j}{\partial x_k}\right|. \tag{2.2.9}$$

Now we can rewrite the arguments of G in I_l and I_r, as follows:

$$\sum_{k=1}^{n-1} X_k(x')\left|\frac{\partial u}{\partial x_k}\right|^2 + X_n(x')\left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial y_j}{\partial \lambda}\right|^{-2}\left\{\sum_{k=1}^{n-1} X_k(x')\left|\frac{\partial y_j}{\partial x_k}\right|^2 + X_n(x')\right\} \tag{2.2.10}$$

on $\gamma_{i,j}$, and

$$\sum_{k=1}^{n-1} X_k(x')\left|\frac{\partial u^s}{\partial x_k}\right|^2 + X_n(x')\left|\frac{\partial u^s}{\partial y}\right|^2 = \left|\frac{\partial y^s}{\partial \lambda}\right|^{-2}\left\{\sum_{k=1}^{n-1} X_k(x')\left|\frac{\partial y^s}{\partial x_k}\right|^2 + X_n(x')\right\} \tag{2.2.11}$$

on D_i^s. In order to prove (2.2.4), it is sufficient to show

$$\sum_{j=1}^{N}\int_{\gamma_{i,j}} F(x', u)\ G\left(\left\{\sum_{k=1}^{n-1} X_k(x')\left|\frac{\partial u}{\partial x_k}\right|^2 + X_n(x')\left|\frac{\partial u}{\partial y}\right|^2\right\}^{\frac{1}{2}}\right)\ dy$$

$$\geq 2\int_{D_i^s \cap \{(x', y)|y \geq 0\}} F(x', u^s)\ G\left(\left\{\sum_{k=1}^{n-1} X_k(x')\left|\frac{\partial u^s}{\partial x_k}\right|^2 + X_n(x')\left|\frac{\partial u^s}{\partial y}\right|^2\right\}^{\frac{1}{2}}\right)\ dy. \tag{2.2.12}$$

for $i = 0, \ldots, M-1$. If we make the change of variables $y \mapsto u(x', y_j)$ in γ_i, j and $y^s \mapsto u^s(x', y^s)$ in $D_i^s \cap \{(x', y) | y \geq 0\}$, we obtain using (2.2.9), (2.2.10) and (2.2.11).

$$\int_{a_i}^{a_{i+1}} F(x', \lambda) \sum_{j=1}^{N} \left\{ \left| \frac{\partial y_j}{\partial \lambda} \right| \; G \left(\left| \frac{\partial y_j}{\partial \lambda} \right|^{-1} \left\{ \sum_{k=1}^{n-1} X_k(x') \left| \frac{\partial y_j}{\partial x_k} \right|^2 + X_n(x') \right\}^{\frac{1}{2}} \right) \right\} \, d\lambda$$
$$\geq \int_{a_i}^{a_{i+1}} F(x', \lambda) \left(\sum_{j=1}^{N} \left| \frac{\partial y_j}{\partial \lambda} \right| \right)$$
$$G \left(2 \left(\sum_{j=1}^{N} \left| \frac{\partial y_j}{\partial \lambda} \right| \right)^{-1} \left\{ \frac{1}{4} \sum_{k=1}^{n-1} X_k(x') \left| \sum_{j=1}^{N} (-1)^j \frac{\partial y_j}{\partial x_k} \right|^2 + X_n(x') \right\}^{\frac{1}{2}} \right) d\lambda. \tag{2.2.13}$$

Then (2.2.12) can be obtained by showing

$$\sum_{j=1}^{N} \left\{ \left| \frac{\partial y_j}{\partial \lambda} \right| \; G \left(\left| \frac{\partial y_j}{\partial \lambda} \right|^{-1} \left\{ \sum_{k=1}^{n-1} X_k(x') \left| \frac{\partial y_j}{\partial x_k} \right|^2 + X_n(x') \right\}^{\frac{1}{2}} \right) \right\}$$

$$\geq \left(\sum_{j=1}^{N} \left| \frac{\partial y_j}{\partial \lambda} \right| \right) \; G \left(\left(\sum_{j=1}^{N} \left| \frac{\partial y_j}{\partial \lambda} \right| \right)^{-1} \left\{ \sum_{k=1}^{n-1} X_k(x') \left| \sum_{j=1}^{N} (-1)^j \frac{\partial y_j}{\partial x_k} \right|^2 + 4 X_n(x') \right\}^{\frac{1}{2}} \right). \tag{2.2.14}$$

Consider $\alpha_j = \left| \frac{\partial y_j}{\partial \lambda} \right| \left(\sum_{j=1}^{N} \left| \frac{\partial y_j}{\partial \lambda} \right| \right)^{-1}$, and

$z_j = \left| \frac{\partial y_j}{\partial \lambda} \right|^{-1} \left\{ \sum_{k=1}^{n-1} X_k(x') \left| \frac{\partial y_j}{\partial x_k} \right|^2 + X_n(x') \right\}^{\frac{1}{2}}$. Note that $0 \leq \alpha_j \leq 1$ and $\sum_{j=1}^{N} \alpha_j = 1$. So, by convexity of G,

$$\sum_{j=1}^{N} \alpha_j G(z_j) \geq G \left(\sum_{j=1}^{N} \alpha_j z_j \right). \tag{2.2.15}$$

Since G is nondecreasing, (2.2.14) can be obtained by showing

$$\sum_{j=1}^{N} \left\{ \sum_{k=1}^{n-1} X_k(x') \left| \frac{\partial y_j}{\partial x_k} \right|^2 + X_n(x') \right\}^{\frac{1}{2}} \geq \left\{ \sum_{k=1}^{n-1} X_k(x') \left| \sum_{j=1}^{N} (-1)^j \frac{\partial y_j}{\partial x_k} \right|^2 + 4 X_n(x') \right\}^{\frac{1}{2}}. \tag{2.2.16}$$

If we introduce the notation $a_0^j := \sqrt{X_n(x')}$ and $a_k^j := \sqrt{X_k(x')} \cdot \left|\dfrac{\partial y_j}{\partial x_k}\right|$ for $k = 1, \ldots, n-1$ y $j = 1, \ldots, N$, then the left member of (2.2.16) represents the sum over j of the length of the N vectors $a^j = (a_0^j, \ldots, a_{n-1}^j)$. Applying the Minkowski's inequality we get:

$$
\begin{aligned}
\sum_{j=1}^{N}\left\{\sum_{k=1}^{n-1} X_k(x')\left|\frac{\partial y_j}{\partial x_k}\right|^2 + X_n(x')\right\}^{\frac{1}{2}} &= \sum_{j=1}^{N}\left(\sum_{k=0}^{n-1}(a_k^j)^2\right)^{\frac{1}{2}} \\
&\geq \left(\sum_{k=0}^{n-1}\left(\sum_{j=1}^{N} a_k^j\right)^2\right)^{\frac{1}{2}} \\
&= \left\{n^2 X_n(x') + \sum_{k=1}^{n-1}\left(\sqrt{X_k(x')}\sum_{j=1}^{N}\left|\frac{\partial y_j}{\partial x_k}\right|\right)^2\right\}^{\frac{1}{2}} \\
&\geq \left\{4X_n(x') + \sum_{k=1}^{n-1} X_k(x')\left|\sum_{j=1}^{N}(-1)^j\frac{\partial y_j}{\partial x_k}\right|^2\right\}^{\frac{1}{2}}
\end{aligned}
\tag{2.2.17}
$$

Since $n \geq 2$. This proves (2.2.16).

Assuming the hypotesis in (b), the equality holds only under the following conditions: $N(i, x') = 2$ for $i = 0, \ldots, M-1$ and and almost every $x' \in \Omega'$, $\frac{\partial y_1}{\partial x_k} = -\frac{\partial y_2}{\partial x_k}$ and $\frac{\partial y_1}{\partial \lambda} = -\frac{\partial y_2}{\partial \lambda}$ for almost every $x' \in \Omega'$ and λ, and $k = 1, \ldots, n$, where the last equality is obtained of (2.2.15), using the strictly convexity of G. These conditions imply that for almost every $x' \in \Omega'$ and λ, $\nabla(y_1 + y_2) = 0$, so that $y_1 + y_2$ is constant, and assuming that $y_1(x', 0) = -\omega$ and $y_2(x', 0) = \omega$ for $x' \in \Omega'$, we have $y_1 = -y_2$. In this way, for such x', λ, and as $y_2 > 0$ in (2.2.7), get $y^s(x', \lambda) = y_2(x', \lambda)$. Thus, by definition of y^s and y_j, obtain $u(x', y_2) = u^s(x', y_2)$ and $u(x', y_1) = u^s(x', y_1)$ a.e. in $\overline{\Omega}$. Moreover, since u is Lipschitz continuous, so u^s (see [16]). Therefore, $u(x', y) = u^s(x', y)$ for all $(x', y) \in \overline{\Omega}$. We conclude that $u = u^s$ if and only if the equality holds in (2.2.3). □

We have the following corollary.

Corollary 2.2.14. *Let* $1 < p < \infty$. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with* Ω^s *its*

Steiner symmetrization with respect to a hyperplane. Let $u \in W_0^{1,p}(\Omega)$ be a non-negative function and let u^s be its Steiner symmetrization. Then

$$\int_{\Omega^s} |\nabla u^s|^p dx \leq \int_{\Omega} |\nabla u|^p dx. \tag{2.2.18}$$

In particular, $u^s \in W_0^{1,p}(\Omega^s)$.

Proof: We assume, without loss of geenerality that the hyperplane of symmetry is $y = 0$. Let Ω' be the projection of Ω on $y = 0$. We assume that $\Omega \subset \hat{\Omega} \times (-w, w)$. In order to prove (2.2.18), one extends by zero the function u so that this is defined in $\hat{\Omega} = \Omega' \times (-w, w)$, where $\Omega' \subset \mathbb{R}^{n-1}$ is a bounded domain. Then we can approximate u by a sequence (u_n) of nice functions such that $u_n = 0$ en $\partial\Omega$, for any n, i.e., each u_n belongs to $W_0^{1,p}(\Omega)$ by (1.2.7). Thus, by Theorem 2.2.13, each u_n is such that

$$\int_{\hat{\Omega}^s} |\nabla u_n^s|^p dx \leq \int_{\hat{\Omega}} |\nabla u_n|^p dx. \tag{2.2.19}$$

The sequence $\{u_n^s\}$ is bounded in $W_0^{1,p}(\Omega^s)$ and we can assume that is weakly convergent to a function v in $W^{1,p}(\Omega^s)$ and hence, by Rellich theorem, $u_n^s \to v$ in $L^p(\Omega^s)$. On the other hand, since $u_n \to u$ in $L^p(\Omega)$, applying Proposition 2.1.11 with $F(s) = s^p$, we have $u_n^s \to u^s$ in $L^p(\Omega^s)$. Therefore $u^s = v$. Since, u_n^s converges weakly to u^s in $W_0^{1,p}(\Omega^S)$ and u_n converges strongly to u in $W_0^{1,p}(\Omega)$, using the lower semi-continuity of the $W^{1,p}$ norm for the weak convergence, we get

$$\int_{\Omega^s} |\nabla u^s|^p dx \leq \liminf \int_{\Omega^s} |\nabla u_n^s|^p dx \leq \liminf \int_{\Omega} |\nabla u_n|^p dx = \int_{\Omega} |\nabla u|^p dx.$$

□

Now we enunciate the theorem which allow us to establish the conditions that a domain Ω and a function u defined in Ω must satisfy for the equality to hold in (2.2.18).

Let $f : \mathbb{R}^n \to [0, +\infty)$ be a strictly convex function such that $f(0) = 0$ and

$$f(\xi_1, \ldots, \xi_{n-1}, \xi_n) = f(\xi_1, \ldots, \xi_{n-1}, -\xi_n), \text{ for every } (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{2.2.20}$$

Also consider $\Omega \subset \mathbb{R}^n$ an open set satisfying:

1.

$$\Omega' \text{ is connected} \tag{2.2.21}$$

where Ω' denote the orthogonal projection of Ω on $\mathbb{R}^{n-1}$.

2.

$$\Omega \text{ is bounded in the direction } y. \tag{2.2.22}$$

3.

$$\Omega \text{ has a locally finite perimeter in } \Omega' \times \mathbb{R} \tag{2.2.23}$$

and

4.

$$H^{n-1}\left(\{(x',y) \in \partial^*\Omega : n_y(x',y) = 0\} \cap (\Omega' \times \mathbb{R})\right) = 0, \tag{2.2.24}$$

where n_y is the component along the y -axis of inner normal vector n to Ω and H^k is the Hausdorff k- dimensional measure. In simple words, it must be satisfied that the measure theoretic boundary $\partial^*\Omega$ is not parallel to y- axis a.e. inside the open cylinder $\Omega' \times \mathbb{R}$.

Let u be a function belonging to

$$W^{1,1}_{0,y}(\Omega) = \{u : \Omega \to \mathbb{R} : u_0 \in W^{1,1}(\omega \times \mathbb{R}) \text{ for every open set } \omega \subset\subset \Omega'\}$$

where u_0 is the extension of u to $\mathbb{R}^n$, which vanishes outside of Ω. Such a function u must satisfy

$$|(\{(x',y) \in \Omega : \nabla_y u(x',y) = 0\} \cap \{(x',y) \in \Omega : M(x') = 0 \vee u(x',y) < M(x')\})|_n = 0. \tag{2.2.25}$$

where $M(x') = \inf\{t > 0 : \mu_u(x',t) = 0\}$ and $\mu_u(x',t) = L^1\left(\{y \in \mathbb{R} : u_0(x',y) > t\}\right)$.

Theorem 2.2.15. *Let $f : \mathbb{R}^n \to [0,+\infty)$ be a strictly convex function such that $f(0) = 0$ and satisfies (2.2.20). Let Ω be an open subset of $\mathbb{R}^N$ satisfying (2.2.21)-(2.2.24). Let u be a non-negative function in $W^{1,1}_{0,y}(\Omega)$ which satisfies (2.2.25). If*

$$\int_{\Omega^s} f(\nabla u^s)dx = \int_\Omega f(\nabla u)dx < \infty, \tag{2.2.26}$$

Then u is equivalent to u^s modulo translation.

Evidently, if $f(\xi) = |\xi|^p$, with $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n$, the conditions of Theorem 2.2.15 hold. Thus, we obtain the following corollary.

Theorem 2.2.16. *Let Ω be a open bounded subset of $\mathbb{R}^n$ satisfying (2.2.21)-(2.2.24). Let u a non-negative function in $W_0^{1,p}(\Omega)$ satisfying (2.2.25). If*

$$\int_{\Omega^s} |\nabla u^s|^p dx = \int_{\Omega} |\nabla u|^p dx < \infty, \tag{2.2.27}$$

then u is equivalent to u^s modulo translation.

In simple words, the above corollary says that if the equality holds in (2.2.27) and the condition (2.2.25) is satisfied, then Ω is symmetric with respect to hyperplane $y = 0$.

2.3 Derivative of a variational problem

This section is based on Garcia et. al [11]. The following theorem proposes an abstract result about differentiability of variational problems, which allows us to find the derivative of λ_1 with respect to variations of a given domain Ω with $C^{2,\alpha}$ boundary, i.e. an expression which allows us to establish the variation of λ_1 for "small" perturbations of boundary $\partial\Omega$.

Theorem 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$ boundary . Let $A = A(x, \delta, \xi) \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $B = B(x, \delta, \xi) \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Assume that the variational problem*

$$\lambda(\delta) = \inf_{u \in W_0^{1,p}(\Omega)} J_\delta(u), \tag{2.3.1}$$

where

$$J_\delta(u) = \frac{\int_\Omega A(x, \delta, \nabla u)\ dx}{\int_\Omega B(x, \delta, u)\ dx},$$

admits, for $|\delta| < \varepsilon$, a unique solution $u = u_\delta$ satisfying

$$\int_\Omega B(x, \delta, u_\delta)\ dx = 1.$$

We assume that $u_\delta \to u_0$ *in* $C^1(\overline{\Omega})$ *as* $\delta \to 0^+$, *where* u_0 *is a solution of the problem for* $\delta = 0$. *Then the function* $\delta \to \lambda(\delta)$ *is differentiable in* $\delta = 0$, *and moreover*

$$\lambda'(0) = \int_\Omega (A_1(x, 0, \nabla u_0) - \lambda_0 B_1(x, 0, u_0)) \ dx \tag{2.3.2}$$

where $A_1 = \partial_\delta A$ *and* $B_1 = \partial_\delta B$.

Proof: In virtue of the characterization (2.3.1), we have

$$J_\delta(u_\delta) - J_0(u_\delta) \leq \lambda(\delta) - \lambda(0) \leq J_\delta(u_0) - J_0(u_0). \tag{2.3.3}$$

As A, B are differentiable at $\delta = 0$, for small δ, we have

$$A(x, \delta, \xi) = A(x, 0, \xi) + \delta A_1(x, 0, \xi) + A_2(x, \delta, \xi) \tag{2.3.4}$$

and

$$B(x, \delta, z) = B(x, 0, z) + \delta B_1(x, 0, z) + B_2(x, \delta, z) \tag{2.3.5}$$

where A_2, B_2 are $o(\delta)$ when $\delta \to 0^+$, and

$$\left|\frac{A_2(x, \delta, \xi)}{\delta}\right| \leq C_1, \left|\frac{A_2(x, \delta, \xi)}{\delta}\right| \leq C_2, \text{ for } |\xi| \leq M, \delta < \varepsilon, x \in \overline{\Omega}.$$

As

$$J_\delta(u_0) - J_0(u_0) = \frac{\int_\Omega A(x, \delta, \nabla u_0) \ dx}{\int_\Omega B(x, \delta, u_0) \ dx} - \lambda_0, \tag{2.3.6}$$

replacing (2.3.4) and (2.3.5) in the right member of (2.3.6), we have in virtue of (2.3.3), that

$$\begin{aligned}
\frac{\lambda(\delta) - \lambda(0)}{\delta} &\leq \frac{1}{\delta}\left(\frac{\int_\Omega A(x, \delta, \nabla u_0) \ dx}{\int_\Omega B(x, \delta, u_0) \ dx} - \lambda_0\right) \\
&\leq \frac{1}{\delta}\left(\frac{\int_\Omega (A(x, 0, \nabla u_0) + \delta A_1(x, 0, \nabla u_0) + A_2(x, \delta, \nabla u_0)) \ dx}{\int_\Omega (B(x, 0, u_0) + \delta B_1(x, 0, u_0) + B_2(x, \delta, u_0)) \ dx} - \lambda_0\right) \\
&\leq \frac{\int_\Omega (A_1(x, 0, \nabla u_0) - \lambda_0 B_1(x, 0, u_0)) \ dx}{\int_\Omega B(x, \delta, u_0) \ dx} \\
&\quad + \frac{1}{\int_\Omega B(x, \delta, u_0) \ dx} \int_\Omega \frac{(A_2(x, \delta, \nabla u_0) - \lambda_0 B_2(x, \delta, u_0)) \ dx}{\delta} \\
&\quad + \frac{\int_\Omega (A(x, \delta, \nabla u_0) - \lambda_0 B(x, \delta, u_0)) \ dx}{\delta \int_\Omega B(x, \delta, u_0) \ dx}
\end{aligned}$$

Thus, by the dominated convergence theorem,

$$\limsup_{\delta\to 0^+} \frac{\lambda(\delta)-\lambda_0}{\delta} \le \int_\Omega \left(A_1(x,0,\nabla u_0) - \lambda_0 B_1(x,0,u_0)\right)\, dx. \tag{2.3.7}$$

Moreover

$$\begin{aligned}
\frac{\lambda(\delta)-\lambda(0)}{\delta} &\ge \frac{J_\delta(u_\delta)-J_0(u_\delta)}{\delta}\\
&= \frac{1}{\delta}\left(\int_\Omega A(x,\delta,\nabla u_\delta)\, dx - \frac{\int_\Omega A(x,0,\nabla u_\delta)\, dx}{\int_\Omega B(x,0,u_\delta)\, dx}\right)\\
&= \frac{\int_\Omega A(x,\delta,\nabla u_\delta)\, dx \int_\Omega B(x,0,u_\delta)\, dx - \int_\Omega A(x,0,\nabla u_\delta)\, dx \int_\Omega B(x,\delta,u_\delta)\, dx}{\delta \int_\Omega B(x,0,u_\delta)\, dx}\\
&= \frac{\int_\Omega \left(A(x,\delta,\nabla u_\delta) - A(x,0,\nabla u_\delta)\right)\, dx}{\delta}\\
&\quad -\frac{\int_\Omega \left(B(x,\delta,u_\delta)-B(x,0,u_\delta)\right)\, dx}{\delta}\cdot \frac{\int_\Omega A(x,0,\nabla u_\delta)\, dx}{\int_\Omega B(x,0,u_\delta)\, dx}.
\end{aligned} \tag{2.3.8}$$

Using the mean value theorem,

$$\frac{\int_\Omega \left(A(x,\delta,\nabla u_\delta)\ - A(x,0,\nabla u_\delta)\right)}{\delta}\, dx = \int_\Omega \int_0^1 \frac{\partial}{\partial s} A(x,s\delta,\nabla u_\delta)\, ds. \tag{2.3.9}$$

Thus, taking lower limit when δ tends to 0^+ in (2.3.9) we get

$$\begin{aligned}
\liminf_{\delta\to 0^+} \frac{\int_\Omega \left(A(x,\delta,\nabla u_\delta) - A(x,0,u_\delta)\right)}{\delta} &= \liminf_{\delta\to 0^+} \int_\Omega \int_0^1 \frac{\partial}{\partial s} A(x,0,\nabla u_0)\, ds\\
&= \int_\Omega A_1(x,0,\nabla u_0)\, dx.
\end{aligned} \tag{2.3.10}$$

In a similar way we obtain that

$$\liminf_{\delta\to 0^+} \frac{\int_\Omega \left(B(x,\delta,u_\delta) - \int_\Omega B(x,0,u_\delta)\right)}{\delta}\, dx = \int_\Omega B_1(x,0,u_0)\, dx. \tag{2.3.11}$$

Taking lower limit when δ tends to 0^+ in (2.3.8), using (2.3.10),(2.3.11) and the fact that by hipotesis $u_\delta \to u_0$ in $C^1(\overline{\Omega})$, we have

$$\liminf_{\delta\to 0^+} \frac{\lambda(\delta)-\lambda_0}{\delta} \ge \int_\Omega \left(A_1(x,0,\nabla u_0) - \lambda_0 B_1(x,0,u_0)\right)\, dx \tag{2.3.12}$$

By (2.3.7) and (2.3.12) we obtain the desired conclusion. □

2.4 Derivative of the volume

Let Ω be a domain in $\mathbb{R}^n$. Consider a family of diffeomorphisms $T = T_\delta(x) \in C^1(\overline{\Omega}, \mathbb{R}^n)$, with small $\delta > 0$, such that

$$T_\delta(x) = x + \delta R(x) + S(x, \delta), \tag{2.4.1}$$

where $R, S(\cdot, \delta) \in C^1(\overline{\Omega}, \mathbb{R}^n), S(x, \delta) = o(\delta)$ when $\delta \to 0$ in $C^1(\overline{\Omega}, \mathbb{R}^n)$.
Let $V(\cdot)$ be the volume functional that maps every domain to its volume. In this case denote $T_\delta(\Omega) = \Omega_\delta$ and $V(\Omega_\delta) = V(\delta)$. We shall show that function $\delta \mapsto V(\delta)$ is differentiable in $\delta = 0$. Observe that

$$V(\delta) = \int_{\Omega_\delta} dy. \tag{2.4.2}$$

If we consider $x = T_\delta^{-1}(y)$ in Ω, then $y \approx x + \delta R(x)$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then

$$V(\delta) = \int_\Omega C(x, \delta)\ dx. \tag{2.4.3}$$

with $C(x, \delta) = |det(DT_\delta(x))|$. Thus, we introduce the next theorem, which gives us an expression for the derivative of the volume functional for small perturbations of a fixed domain whose boundary is $C^{2,\alpha}$.

Theorem 2.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$ boundary, $\Omega_\delta = T_\delta(\Omega)$ the perturbation of Ω asociated to a family $T_\delta(x) = x + \delta R(x) + S(x, \delta)$. Then the volume $V(\delta) = V(\Omega_\delta)$ is differentiable with respect to δ at $\delta = 0$. Moreover*

$$V'(0) = \int_{\partial\Omega} \langle R, n \rangle\ d\sigma, \tag{2.4.4}$$

where n is the outward unit normal vector field on $\partial\Omega$.

Proof: We note that

$$\frac{V(\delta) - V(0)}{\delta} = \frac{\displaystyle\int_\Omega C(x, \delta)\ dx - V(0)}{\delta}. \tag{2.4.5}$$

Further

$$C(x,\delta) = 1 + \delta \mathrm{div} R(x) + C_2(x,\delta) \tag{2.4.6}$$

where C_2 is $o(\delta)$ when $\delta \to 0^+$, and

$$\left| \frac{C_2(x,\delta)}{\delta} \right| \leq C, \text{ for } \delta < \varepsilon, x \in \overline{\Omega}.$$

Thus, replacing (2.4.6) in (2.4.5) we have

$$\begin{aligned} \frac{V(\delta) - V(0)}{\delta} &= \frac{\int_\Omega (1 + \delta \mathrm{div} R(x) + C_2(x,\delta)) \; dx - \int_\Omega dx}{\delta} \\ &= \int_\Omega \mathrm{div} R(x) \; dx + \int_\Omega \frac{C_2(x,\delta)}{\delta} \; dx. \end{aligned}$$

Taking limit as $\delta \to 0^+$ it gives

$$V'(0) = \lim_{\delta \to 0^+} \frac{V(\delta) - V(0)}{\delta} = \int_\Omega \mathrm{div} R(x) \; dx,$$

and by divergence theorem

$$V'(0) = \int_{\partial\Omega} \langle R, n \rangle \; d\sigma.$$

. □

Chapter 3

Properties related to eigenvalues of the Dirichlet p - Laplacian

We assume throughout this chapter that Ω is a bounded domain in $\mathbb{R}^n$. Given $1 < p < \infty$, the p-Laplacian operator, which is denoted by Δ_p, is defined as

$$\begin{aligned} \Delta_p : W_0^{1,p}(\Omega) &\longrightarrow W^{-1,q}(\Omega) \\ u &\mapsto \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \end{aligned} \tag{3.0.1}$$

where $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$ with p, q conjugate exponent.

3.1 On the Dirichlet p-Laplacian eigenvalue problem

The treatment of this section is based mostly on Lindqvist [19].

Definition 3.1.1. *We say $\lambda \in \mathbb{R}$ is an **eigenvalue for the Dirichlet p- Laplacian** if there exists nonzero $u \in W_0^{1,p}(\Omega)$ with*

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1.1}$$

in the weak sense.

We have that if (u,λ) is a solution to problem (3.1.1), then for all $\alpha\in\mathbb{R}, (\alpha u,\lambda)$ is also a solution. Thus, one can find an eigenfunction with $\|u\|_p = 1$.

If u is an eigenfunction of λ, using u as a test function in (3.1.1), we obtain

$$\lambda = \frac{\int_\Omega |\nabla\varphi|^p}{\int_\Omega |\varphi|^p}. \tag{3.1.2}$$

The expression $\frac{\int_\Omega |\nabla\varphi|^p}{\int_\Omega |\varphi|^p}$ is called **Rayleigh's quotient** at $\varphi\in W_0^{1,p}(\Omega)$. It is deduced that any eigenvalue of this problem is non-negative.

Proposition 3.1.2. *Every eigenfunction corresponding to an arbitrary eigenvalue λ belongs to $C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha>0$.*

Proof: This may be deduced from the regularity theory for quasilinear elliptic equations. See [12],[13] and [25]. □

Let R be the functional defined by nonzero functions in $W_0^{1,p}(\Omega)$ by its Rayleigh's quotient, i.e.

$$R(u) = \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}. \tag{3.1.3}$$

We make the following important observation.

Proposition 3.1.3. *There exists an equivalence between the eigenvalue problem (3.1.1) and the Euler Lagrange equation at critical points of the functional R, defined in (3.1.3), on $W_0^{1,p}(\Omega)\setminus\{0\}$ in the following sense. Every eigenvalue is a critical value of the functional R and u is an eigenfunction if and only if u is a critical point of the functional R.*

Proof: It follows from Lemma 3.4 (i) in [2] that R has a Gateaux derivative on $W_0^{1,p}(\Omega)\setminus\{0\}$. It can be seen that

$$\nabla_G R(u) = 0 \Leftrightarrow \int_\Omega |\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle = R(u)\int_\Omega |u|^{p-2}u\varphi, \text{ for all } \varphi\in W_0^{1,p}(\Omega). \tag{3.1.4}$$

Thus, clearly if u is a critical point of R, then $R(u)$ is an eigenvalue and u an eigenfunction. Reciprocally, if u an eigenfunction associated to eigenvalue λ, then by

(3.1.2), $R(u) = \lambda$. It follows from the weak formulation of (3.1.1) and (3.1.4) that u is a critical point of R. □

Let

$$\lambda_1(\Omega) = \inf_{\varphi \in W_0^{1,p}(\Omega), \varphi \neq 0} \frac{\int_\Omega |\nabla \varphi|^p}{\int_\Omega |\varphi|^p}. \tag{3.1.5}$$

By the above proposition, if this infimum is attained, then it is an eigenvalue of the problem (**??**), since this would be a critical value of R. Moreover, $\lambda_1(\Omega)$ would be the least eigenvalue, since $\lambda_1(\Omega)$ would be the minimal critical value and every eigenvalue is a critical point of R. Indeed, this fact is proved in the following proposition.

Proposition 3.1.4. *The infimum of the Rayleigh's quotient in (3.1.5) is attained.*

Proof: Let $(u_n)_{n=1}^\infty$ be a minimizing sequence for the functional R. We can assume that $\displaystyle\int_\Omega |u_n|^p = 1$ for all n. So $(u_n)_{n=1}^\infty$ is bounded in $W_0^{1,p}(\Omega)$ and without loss of generality, we can assume that this sequence converges weakly to $u \in W_0^{1,p}(\Omega)$. Further, by the Rellich Kondrasov Theorem (1.2.8), (u_n) converges strongly to u in $L^p(\Omega)$. This gives,

$$\int_\Omega |u|^p = \lim_{n\to\infty} \int_\Omega |u_n|^p = 1.$$

By the lower semicontinuity of the $W_0^{1,p}$ norm, we have

$$\lambda_1(\Omega) = \lim_{n\to\infty} \int_\Omega |\nabla u_n|^p \geq \int_\Omega |\nabla u|^p \geq \lambda_1(\Omega).$$

This shows that $\lambda_1(\Omega)$ is attained. □

By the above proposition, the first eigenvalue is characterized as

$$\lambda_1(\Omega) = \min_{\varphi \in W_0^{1,p}(\Omega), \varphi \neq 0} \frac{\int_\Omega |\nabla \varphi|^p}{\int_\Omega |\varphi|^p}. \tag{3.1.6}$$

This proposition also shows that there exists an eigenfunction corresponding to λ_1 . We will say "first eigenfunction" for an eigenfunction associated to λ_1.

Proposition 3.1.5. *Let C_0 be the least positive constant such that*

$$\int_\Omega |\varphi|^p \leq C \int_\Omega |\nabla \varphi|^p, \ \textit{for every } \varphi \in W_0^{1,p}(\Omega), \tag{3.1.7}$$

then $\frac{1}{C_0} = \lambda_1$.

Proof: We note that the such constant C_0 exists by Poincare's inequality. So,

$$\frac{1}{C_0} \leq \frac{\int_\Omega |\nabla\varphi|^p}{\int_\Omega |\varphi|^p},$$

for every nonzero function $\varphi \in W_0^{1,p}(\Omega)$. Thus, by definition of λ_1, we get $\frac{1}{C_0} \leq \lambda_1$. Anew by definition of λ_1 we observe that

$$\int_\Omega |\varphi|^p \leq \frac{1}{\lambda_1}\int_\Omega |\nabla\varphi|^p, \text{ for every } \varphi \in W_0^{1,p}(\Omega),$$

i.e. $\frac{1}{\lambda_1}$ satisfy (3.1.7). Therefore $C_0 \leq \dfrac{1}{\lambda_1}$, which together with the previous observation gives $\frac{1}{C_0} = \lambda_1$. □

Let us look at the effect of a translation to Ω on an arbitrary eigenvalue λ. Let us denote by τ_{x_0} the translation by x_0, i.e. if $x \in \Omega$, then $\tau_{x_0}(x) = x + x_0$. If u is a function defined in Ω, we define the function $\tau_{x_0}u$ in $\tau_{x_0}(\Omega)$ by the formula $\tau_{x_0}u(x) = u(x - x_0)$. We have the following proposition.

Proposition 3.1.6. *Let Ω be an arbitrary domain and $x_0 \in \mathbb{R}^n$. The value λ is an eigenvalue of Ω if and only if λ is an eigenvalue of $\tau_{x_0}(\Omega)$.*

Proof: It is sufficient to show that if λ is an eigenvalue of Ω, then it is an eigenvalue of $\tau_{x_0}(\Omega)$. Let u be an eigenfunction of λ on Ω. Let $v = \tau_{x_0}u$ and we consider $\hat{\varphi}(x) = \tau_{x_0}\varphi(x)$, with $\varphi \in W_0^{1,p}(\Omega)$. We note that $\nabla v(x) = \nabla u(x - x_0)$ and

$\nabla\hat{\varphi}(x) = \nabla\varphi(x - x_0)$. Then

$$\begin{aligned}
&\int_{\tau_{x_0}(\Omega)} |\nabla v(x)|^{p-2}\langle\nabla v(x), \nabla\hat{\varphi}(x)\rangle \ dx \\
&= \int_{\tau_{x_0}(\Omega)} |\nabla u\,(x - x_0)\,|^{p-2}\langle\nabla u\,(x - x_0)\,, \nabla\varphi\,(x - x_0)\rangle \ dx \\
&= \int_{\Omega} |\nabla u\,(y)\,|^{p-2}\langle\nabla u\,(y)\,, \nabla\varphi\,(y)\rangle \ dy \\
&= \lambda\int_{\Omega} |u\,(y)|^{p-2}u\,(y)\,\varphi\,(y) \ \ dy \\
&= \lambda\int_{\tau_{x_0}(\Omega)} |u\,(x - x_0)|^{p-2}u\,(x - x_0)\,\varphi\,(x - x_0) \ \ dx \\
&= \lambda\int_{\tau_{x_0}(\Omega)} |v(x)|^{p-2}v(x)\hat{\varphi}(x)dx.
\end{aligned} \tag{3.1.8}$$

Thus, by definition of an eigenvalue on $(\tau_{x_0}(\Omega))$ we get the desired conclusion. □

Corollary 3.1.7. *Let Ω be an arbitrary domain. We have*

$$\lambda_1(\tau_{x_0}(\Omega)) = \lambda_1(\Omega). \tag{3.1.9}$$

for any $x_0 \in \mathbb{R}^n$.

Proof: Let u be an eigenfunction for $\lambda_1(\Omega)$. By Proposition 3.1.13 which is proved further down, we may assume that $u > 0$. We know by the previous proposition that $\lambda_1(\Omega)$ is also an eigenvalue of $\tau_{x_0}(\Omega)$. But, indeed from the proof we have $v = \tau_{x_0}u$ is an eigenfunction of $\lambda_1(\Omega)$ as eigenvalue on $\tau_{x_0}(\Omega)$. Since $u > 0$ in Ω, it follows by the definition of v that $v > 0$ in $\tau_{x_0}(\Omega)$. By this and by Proposition 3.1.17, proved further down, we obtain (3.1.9). □

Let us also look at the effect of homothety . Let $O \in \Omega, k > 0$ and H_k be homothety of Ω about the origin O by the factor k, i.e. if $x \in \Omega$, then $H_k(x) = kx$. If u is a function defined in Ω, we define the function $H_k u$ in $H_k(\Omega)$ by the formula $H_k u(x) = u(\frac{x}{k})$.

Proposition 3.1.8. *Let Ω be an arbitrary domain. If λ is an eigenvalue of Ω, then $\dfrac{\lambda}{k^p}$ is an eigenvalue of $H_k(\Omega)$.*

Proof: Let u be an eigenfunction of λ. Let $v = H_k u$ and we consider $\hat{\varphi}(x) = H_k\varphi(x)$, with $\varphi \in W_0^{1,p}(\Omega)$. We note that $\nabla v(x) = \frac{1}{k}\nabla u(\frac{x}{k})$ and $\nabla\hat{\varphi}(x) = \frac{1}{k}\nabla\varphi(\frac{x}{k})$. Then

$$\begin{aligned}\int_{H_k(\Omega)} |\nabla v(x)|^{p-2}\langle\nabla v(x), \nabla\hat{\varphi}(x)\rangle\, dx &= \frac{1}{k^p}\int_{H_k(\Omega)} \left|\nabla u\left(\frac{x}{k}\right)\right|^{p-2} \left\langle \nabla u\left(\frac{x}{k}\right), \nabla\varphi\left(\frac{x}{k}\right)\right\rangle dx \\ &= \frac{1}{k^p}\int_{\Omega} |\nabla u\,(y)\,|^{p-2}\langle\nabla u\,(y)\,, \nabla\varphi\,(y)\rangle\; k^n\; dy \\ &= \frac{\lambda}{k^p}\int_{\Omega} |u\,(y)|^{p-2} u\,(y)\,\varphi\,(y)\;\; k^n\; dy \\ &= \frac{\lambda}{k^p}\int_{H_k(\Omega)} \left|u\left(\frac{x}{k}\right)\right|^{p-2} u\left(\frac{x}{k}\right)\varphi\left(\frac{x}{k}\right)\; dx \\ &= \frac{\lambda}{k^p}\int_{H_k(\Omega)} |v(x)|^{p-2} v(x)\hat{\varphi}(x)\; dx.\end{aligned}$$

This shows that $\dfrac{\lambda}{k^p}$ is an eigenvalue for $H_k(\Omega)$. □

Corollary 3.1.9. *Let Ω be an arbitrary domain. We have*

$$\lambda_1(H_k(\Omega)) = \frac{\lambda_1(\Omega)}{k^p}. \tag{3.1.10}$$

Proof: Let u be an eigenfunction for $\lambda_1(\Omega)$. By Proposition 3.1.13 proved below, we may assume that $u > 0$. We know by the previous proposition that $\dfrac{\lambda_1(\Omega)}{k^p}$ is an eigenvalue of $H_k(\Omega)$. But, indeed from the proof if u is an eigenfunction for $\lambda_1(\Omega)$, then $v = H_k u$ is an eigenfunction of $\dfrac{\lambda_1(\Omega)}{k^p}$ on $\tau_{x_0}(\Omega)$. Since $u > 0$ in Ω, it follows by the definition of v that $v > 0$ in $H_k(\Omega)$. By this and Proposition 3.1.17 we conclude (3.1.10). □

Remark 3.1.10. *An important consequence corresponding to the above results is that if we have two domains Ω, Ω' such that $|\Omega| > |\Omega'|$ and $\lambda_1(\Omega) > \lambda_1(\Omega')$, then we can get a domain Ω'' such that $|\Omega| = |\Omega''|$ and $\lambda_1(\Omega) > \lambda_1(\Omega'')$.*

We now state a boundary regularity result for first eigenfunctions.

Proposition 3.1.11. *Every first eigenfunction belongs to $C^1(\overline{\Omega})$.*

Proof: See Barles [3]. □

The next theorem will be used in the proof of the following property of the first eigenfunction.

Theorem 3.1.12. *(Harnack's inequality) Let B_r and B_{2r} be concentric balls such that $B_{2r} \subset \Omega$. If u is a non-negative eigenfunction then*

$$\sup_{B_r} u \leq C \inf_{B_r} u.$$

Proof: See Lindqvist [20]. □

Proposition 3.1.13. . *There exists a non-negative eigenfunction corresponding to the first eigenvalue. Moreover, any first eigenfunction $\lambda_1(\Omega)$ is either strictly positive or strictly negative in the domain Ω.*

Proof: Let u be an eigenfunction of λ_1. Then, observe that

$$|\nabla|u|| = \left|\frac{u}{|u|}\nabla u\right| = |\nabla u|,$$

Thus, if u minimizes the Rayleigh quotient, then so does $|u|$, so that $|u|$ is also an eigenfunction of λ_1. Let $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r) \subset \Omega$. We suppose that $|u(x_0)| = 0$. By Harnack's inequality, $|u|$ vanishes in $B(x_0, \frac{r}{2})$, and so, the set $\{|u| = 0\}$ is open in $\mathbb{R}^n$. On the other hand, since $|u|$ is continuous in $\overline{\Omega}$, the set $\{|u| = 0\}$ is closed in $\mathbb{R}^n$. Since Ω is connected this implies $\{|u| = 0\}$ is equal to $\emptyset$ or Ω. However, as u is an eigenfunction and so, $u \neq 0$. Therefore $\{|u| = 0\} = \emptyset$. So, $|u| > 0$ in Ω. It now follows that $u > 0$ or $-u > 0$ in Ω, since u is continuous on Ω and Ω is connected. □

By the above theorem, given a first eigenfunction u either $u > 0$ or $u < 0$ in Ω. Thus, without loss of generality, in all that follows we will consider every first eigenfunction as positive in Ω. The next proposition uses the above result in its proof.

Proposition 3.1.14. *Let Ω be a bounded domain with C^2 boundary in $\mathbb{R}^n$. Let u be an eigenfunction of $\lambda_1(\Omega)$. We have*

$$\frac{\partial u_1}{\partial n} < 0 \text{ on } \partial\Omega$$

Proof: We note that as Ω has C^2 boundary, we have that for every $x_0 \in \partial\Omega$, there exists an open ball $B = B_R(z_0) \subset \Omega$ such that $\partial B \cap \partial\Omega = \{x_0\}$ and the outward normal vector to Ω and B coincide in x_0. For $K, \alpha > 0$, we define a function $b : B \to \mathbb{R}$ as $b(x) = K(e^{-\alpha|x-z_0|^2} - e^{-\alpha R^2})$. We note that $b > 0$ in $B_R(z_0)$, $b(x_0) = 0$ and $\dfrac{\partial b}{\partial \eta}(x_0) < 0$. Moreover, it can be shown that

$$-\Delta_p b(x) = -2Ke^{-\alpha|x-z_0|^2}|\nabla b|^{p-2}(2\alpha^2|x-z_0|^2 - n\alpha).$$

Thus, we can choose α large enough and independent of K such that

$$-\Delta_p b \leq 0 \text{ in } B_R(z_0) \setminus B_{\frac{R}{2}}(z_0),$$

since $|x - z_0|$ is bounded below by $\frac{R}{2}$ in $B_R(z_0) \setminus B_{\frac{R}{2}}(z_0)$.

Further, as $u > 0$ are continuous in Ω and $\partial B_{\frac{R}{2}}(z_0)$ is compact in $\mathbb{R}^n$, we obtain that u is bounded below by a positive constant on $\partial B_{\frac{R}{2}}(z_0)$. Moreover, since the factor $e^{-\alpha|x-z_0|^2} - e^{-\alpha R^2}$ in b is a positive constant on $\partial B_{\frac{R}{2}}(z_0)$, we can choose K small enough such that $b \leq u$ in $\partial B_{\frac{R}{2}}(z_0)$. Also, $b(x) = 0$ for all $x \in \partial B_R(z_0)$, while $u \geq 0$ in $\partial B_{\frac{R}{2}}(z_0)$. Therefore, we have

$$b \leq u \text{ in } \partial(B_R(z_0) \setminus B_{\frac{R}{2}}(z_0)),\ -\Delta_p b \leq 0 \text{ and } -\Delta_p u \leq 0 \text{ en } B_R(z_0) \setminus B_{\frac{R}{2}}(z_0).$$

Thus, by the weak comparison principle of Tolksdorff [25], $b \leq u$ in $B_R(z_0) \setminus B_{\frac{R}{2}}(z_0)$. Moreover, as $b(x_0) = u(x_0) = 0$, this implies that $\frac{\partial b}{\partial n}(x_0) \geq \frac{\partial u}{\partial n}(x_0)$. Since $\frac{\partial b}{\partial n}(x_0) < 0$, we obtain that $\frac{\partial u}{\partial n}(x_0) < 0$. Therefore, as $x_0 \in \partial\Omega$ is arbitrary, we have proved that $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$. □

Now we prove the property which says that the first eigenvalue is simple, that is, any two first eigenfunctions are multiples of each other.

Theorem 3.1.15. *The first eigenvalue is simple.*

Proof: Suppose that $u > 0$ and $v > 0$ in Ω are eigenfunctions both corresponding to λ_1. Set

$$u_\varepsilon = u + \varepsilon,\ v_\varepsilon = v + \varepsilon,$$

being $\varepsilon > 0$ a positive parameter. We use the test- functions

$$\eta_1 = \frac{{u_\varepsilon}^p - {v_\varepsilon}^p}{{u_\varepsilon}^{p-1}} \text{ and } \eta_2 = \frac{{v_\varepsilon}^p - {u_\varepsilon}^p}{{v_\varepsilon}^{p-1}}.$$

From the $C^1(\overline{\Omega})$ regularity of u and v (Proposition 3.1.11), both η_1 and η_2 belong to $W_0^{1,p}(\Omega)$.

Then,

$$\nabla\eta_1 = \left\{1 + (p-1)\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^p\right\}\nabla u - p\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{p-1}\nabla v,$$

and, by symmetry, the gradient of $\nabla\eta_2$ has an analogous expression with u and v interchanged.

As u is an eigenfunction corresponding to λ_1,

$$\int_\Omega |\nabla u|^{p-2}\langle\nabla u, \nabla\varphi\rangle\ dx = \lambda_1 \int_\Omega u^{p-1}\varphi\ dx, \tag{3.1.11}$$

and analogously for v. Inserting the function η_1 in (3.1.11), the function η_2 in the respective expression for v and adding these expressions, we obtain the expression

$$\begin{aligned}
&\lambda_1 \int_\Omega \left[\left(\frac{u}{u_\varepsilon}\right)^{p-1} - \left(\frac{v}{v_\varepsilon}\right)^{p-1}\right](u_\varepsilon^p - v_\varepsilon^p)\ dx \\
&= \int_\Omega \left[\left\{1 + (p-1)\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^p\right\}|\nabla u_\varepsilon|^p + \left\{1 + (p-1)\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^p\right\}|\nabla v_\varepsilon|^p\right]\ dx \\
&\quad - \int_\Omega \left[p\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{p-1}|\nabla u_\varepsilon|^{p-2}\langle\nabla u_\varepsilon, \nabla v_\varepsilon\rangle - p\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{p-1}|\nabla v_\varepsilon|^{p-2}\langle\nabla v_\varepsilon, \nabla u_\varepsilon\rangle\right]\ dx \\
&= \int_\Omega \left[\{{u_\varepsilon}^p + (p-1){v_\varepsilon}^p\}|\nabla \ln u_\varepsilon|^p + \{{v_\varepsilon}^p + (p-1){u_\varepsilon}^p\}|\nabla \ln v_\varepsilon|^p\right]\ dx \\
&\quad - \int_\Omega \left[p{v_\varepsilon}^p|\nabla \ln u_\varepsilon|^{p-2}\langle\nabla \ln u_\varepsilon, \nabla \ln v_\varepsilon\rangle + p{u_\varepsilon}^p|\nabla \ln v_\varepsilon|^{p-2}\langle\nabla \ln v_\varepsilon, \nabla \ln u_\varepsilon\rangle\right]\ dx \\
&= \int_\Omega \left[u_\varepsilon^p\left(|\nabla \ln u_\varepsilon|^p - |\nabla \ln v_\varepsilon|^p\right) - pu_\varepsilon^p|\nabla \ln v_\varepsilon|^{p-2}\langle\nabla \ln v_\varepsilon, (\nabla \ln u_\varepsilon - \nabla \ln v_\varepsilon)\rangle\right] \\
&\quad + \left[v_\varepsilon^p\left(|\nabla \ln u_\varepsilon|^p - |\nabla \ln v_\varepsilon|^p\right) - pv_\varepsilon^p|\nabla \ln u_\varepsilon|^{p-2}\langle\nabla \ln u_\varepsilon, (\nabla \ln v_\varepsilon - \nabla \ln u_\varepsilon)\rangle\right]\ dx
\end{aligned} \tag{3.1.12}$$

Moreover, by Lebesgue dominated convergence theorem

$$\lim_{\varepsilon\to 0^+}\int_\Omega \left[\left(\frac{u}{u_\varepsilon}\right)^{p-1} - \left(\frac{v}{v_\varepsilon}\right)^{p-1}\right](u_\varepsilon^p - v_\varepsilon^p) = 0. \tag{3.1.13}$$

Letting ε tend toward zero in (3.1.12) we obtain

$$\begin{aligned} 0 = \int_\Omega & \left[u^p\left(|\nabla \ln u|^p - |\nabla \ln v|^p\right) - pu^p|\nabla \ln v|^{p-2}\langle \nabla \ln v, (\nabla \ln u - \nabla \ln v)\rangle\right] \\ &+ \left[v^p\left(|\nabla \ln u|^p - |\nabla \ln v|^p\right) - pv^p|\nabla \ln u|^{p-2}\langle \nabla \ln u, (\nabla \ln v - \nabla \ln u)\rangle\right]\, dx, \end{aligned} \tag{3.1.14}$$

By Proposition 1.5.2 the integrand in (3.1.14) is non-negative and since u, v are C^1 in $\overline{\Omega}$, we get

$$\begin{aligned} &\left[u^p\left(|\nabla \ln u|^p - |\nabla \ln v|^p\right) - pu^p|\nabla \ln v|^{p-2}\langle \nabla \ln v, (\nabla \ln u - \nabla \ln v)\rangle\right] \\ &+ \left[v^p\left(|\nabla \ln u|^p - |\nabla \ln v|^p\right) - pv^p|\nabla \ln u|^{p-2}\langle \nabla \ln u, (\nabla \ln v - \nabla \ln u)\rangle\right] = 0 \end{aligned}$$

in Ω. Thus, since the equality holds in (1.5.2) if and only if $\omega_1 = \omega_2$, we have that $\nabla \ln u = \nabla \ln v$. It follows that $\nabla\left(\ln(\frac{u}{v})\right) = 0$, and that $\frac{u}{v}$ must be constant in Ω, i.e. there exists $k \in \mathbb{R}$ such that $u = kv$. □

As a consequence of the above theorem we have the next proposition.

Proposition 3.1.16. *Let $\Omega, \hat{\Omega}$ be bounded domains in $\mathbb{R}^n$ such that $\Omega \subset \hat{\Omega}$. Then,*

$$\lambda_1(\Omega) \geq \lambda_1(\hat{\Omega}).$$

Moreover if $\Omega \subset \hat{\Omega}$ in a strict way, then

$$\lambda_1(\Omega) > \lambda_1(\hat{\Omega}).$$

Proof: If $\Omega \subset \hat{\Omega}$, evidently $W_0^{1,p}(\Omega) \subset W_0^{1,p}(\hat{\Omega})$. Thus

$$\lambda_1(\Omega) = \min_{\varphi\in W_0^{1,p}(\Omega),\varphi\neq 0} \frac{\int_\Omega |\nabla \varphi|^p}{\int_\Omega |\varphi|^p} \geq \min_{\hat{\varphi}\in W_0^{1,p}(\hat{\Omega}),\hat{\varphi}\neq 0} \frac{\int_{\hat{\Omega}} |\nabla \hat{\varphi}|^p}{\int_{\hat{\Omega}} |\hat{\varphi}|^p} = \lambda_1(\hat{\Omega}).$$

We suppose that $\Omega \subset \hat{\Omega}$ in strict way. Let $u, \hat{u}$ be the normalized first eigenfunctions in $\Omega, \hat{\Omega}$ respectively. Suppose that $\lambda_1(\Omega) = \lambda_1(\hat{\Omega})$. Let u_0 be the extension by zero of u to $\hat{\Omega}$. As the Rayleigh's quotient of u in Ω is equal to $\lambda_1(\Omega)$, then

$$\int_{\hat{\Omega}} |\nabla u_0|^p = \int_\Omega |\nabla u|^p = \lambda_1(\Omega) = \lambda_1(\hat{\Omega})$$

Thus, u_0 is an eigenfunction corresponding to $\lambda_1(\hat{\Omega})$. Hence, by the above theorem, there exists $k \neq 0$ such that $u_0 = k\hat{u}$ in $\hat{\Omega}$. As $u_0 = 0$ on $\partial\Omega$, and $\hat{u} > 0$ in $\hat{\Omega}$ and $\partial\Omega$ is contained in $\overline{\hat{\Omega}}$, it follows that $\partial\Omega \subset \partial\hat{\Omega}$. By Proposition 1.5.1 it follows that $\Omega = \hat{\Omega}$ □

The following theorem says that only λ_1 has positive eigenfunctions.

Theorem 3.1.17. *If $v > 0$ is an eigenfunction corresponding to the eigenvalue λ, then $\lambda = \lambda_1$.*

Proof: By definition $\lambda_1 \leq \lambda$. Let $u > 0$ be a first eigenfunction. Setting

$$u_\varepsilon = u + \varepsilon, \; v_\varepsilon = v + \varepsilon.$$

We obtain in a similar way as in the above theorem that

$$\int_\Omega \left[\lambda_1 \left(\frac{u}{u_\varepsilon} \right)^{p-1} - \lambda \left(\frac{v}{v_\varepsilon} \right)^{p-1} \right] (u_\varepsilon^p - v_\varepsilon^p) \, dx \geq 0. \tag{3.1.15}$$

Letting ε tend toward zero in (3.1.15), we arrive to

$$(\lambda_1 - \lambda) \int_\Omega (u^p - v^p) \geq 0.$$

Considering that v can be replaced by kv for any k,

$$(\lambda_1 - \lambda) \int_\Omega (u^p - k^p v^p) \geq 0.$$

If $\lambda_1 < \lambda$, then

$$\int_\Omega u^p \leq \int_\Omega k^p v^p.$$

Letting k tend to zero, then $\int_\Omega u^p \leq 0$, and we obtain that $u = 0$ on Ω, a contradiction to the fact $u > 0$ in Ω. □

Next we will dedicate our efforts to prove a result which says that if u is an eigenfunction corresponding to an arbitrary eigenvalue, then the Lebesgue measure of the set of critical points of u, $\{x \in \Omega : \nabla u(x) = 0\}$, is zero. We have the following theorem.

Theorem 3.1.18. *Suppose that* $u \in W^{1,p}(\Omega)$ *is a weak solution of (1.2.5), with* $f \in L^{\gamma}(\Omega)$, *where*

$$\gamma > \frac{n}{p}, \ \gamma \geq 2.$$

Then

$$f(x) = 0 \text{ for almost every } x \text{ in } \{\nabla u = 0\}.$$

Proof: By Lemma 2.1 in [21], $|\nabla u|^{p-1} \in W^{1,2}_{loc}(\Omega)$. Thus, for $\varepsilon > 0$,

$$\frac{|\nabla u|^{p-1}}{\varepsilon + |\nabla u|^{p-1}} \in W^{1,2}_{loc}(\Omega).$$

Therefore, for every $\varphi \in D(\Omega)$, using $\frac{|\nabla u|^{p-1}}{\varepsilon+|\nabla u|^{p-1}}\varphi$ as a test function in (1.2.6), we have

$$\begin{aligned}
\int_{\Omega\setminus\{\nabla u=0\}} \frac{|\nabla u|^{p-1}}{\varepsilon + |\nabla u|^{p-1}}\varphi\ f &= \int_{\Omega} \frac{|\nabla u|^{p-1}}{\varepsilon + |\nabla u|^{p-1}}\varphi\ f \\
&= \int_{\Omega} |\nabla u|^{p-2} \left\langle \nabla u, \nabla\left(\frac{|\nabla u|^{p-1}}{\varepsilon + |\nabla u|^{p-1}}\varphi\right)\right\rangle \\
&= \int_{\Omega\setminus\{\nabla u=0\}} |\nabla u|^{p-2} \frac{|\nabla u|^{p-1}}{\varepsilon + |\nabla u|^{p-1}} \langle \nabla u, \nabla\varphi\rangle \\
&+ \int_{\Omega\setminus\{\nabla u=0\}} |\nabla u|^{p-2}\varphi \frac{\varepsilon}{\left(\varepsilon + |\nabla u|^{p-1}\right)^2} \left\langle \nabla\left(|\nabla u|^{p-1}\right)\ \nabla u\right\rangle .
\end{aligned} \tag{3.1.16}$$

As $\varphi \in D(\Omega)$, there exists $M > 0$ such that $|\nabla\varphi| \leq M$. Moreover,

$$\frac{|\nabla u|^{p-1}}{\varepsilon + |\nabla u|^{p-1}} \leq 1,$$

by this and Holder's inequality we get

$$\left| |\nabla u|^{p-2} \frac{|\nabla u|^{p-1}}{\varepsilon + |\nabla u|^{p-1}} \langle \nabla u, \nabla\varphi\rangle \right| \leq M|\nabla u|^{p-1} \tag{3.1.17}$$

with $M|\nabla u|^{p-1}$ independent of ε belongs to $L^1(\Omega)$.

Further, since $\frac{|\nabla u|^{p-1}}{\varepsilon+|\nabla u|^{p-1}}, \frac{\varepsilon}{\varepsilon+|\nabla u|^{p-1}} \leq 1$, using Cauchy Schwarz inequality in $\mathbb{R}^n$ we obtain

$$\left| |\nabla u|^{p-2}\varphi \frac{\varepsilon}{\left(\varepsilon+|\nabla u|^{p-1}\right)^2} \left\langle \nabla\left(|\nabla u|^{p-1}\right), \nabla u\right\rangle \right|$$

$$\leq \frac{|\nabla u|^{p-1}}{\varepsilon+|\nabla u|^{p-1}} \frac{\varepsilon}{\varepsilon+|\nabla u|^{p-1}} \left|\varphi\nabla\left(|\nabla u|^{p-1}\right)\right| \leq |\varphi\nabla\left(|\nabla u|^{p-1}\right)|, \tag{3.1.18}$$

with $|\varphi\nabla\left(|\nabla u|^{p-1}\right)|$ independent of ε and belongs to $L^1(\Omega)$.

Thus, on the basis of (3.1.17) and (3.1.18), we can use Lebesgue dominated convergence theorem, letting ε tends to 0^+ in (3.1.16) and we get

$$\begin{aligned}\int_{\Omega\setminus\{\nabla u=0\}} \varphi\ f &= \int_{\Omega\setminus\{\nabla u=0\}} |\nabla u|^{p-2}\langle\nabla u, \nabla\varphi\rangle \\ &= \int_{\Omega} |\nabla u|^{p-2}\langle\nabla u, \nabla\varphi\rangle \\ &= \int_{\Omega} \varphi\ f,\ \forall\varphi\in D(\Omega).\end{aligned}$$

Therefore

$$\int_{\{\nabla u=0\}} \varphi\ f = 0,\ \forall\varphi\in D(\Omega).$$

This implies that

$$f(x) = 0 \text{ a.e. in } \{\nabla u = 0\}.$$

□

As a corollary of Theorem 3.1.18 we have the next result.

Corollary 3.1.19. *Under the assumptions of Theorem 3.1.18, if $f(x) \neq 0$ almost everywhere in Ω, then the Lebesgue measure of $\{\nabla u = 0\}$ is zero. Thus, the Lebesgue measure of the level set $\{u = c\}$ is zero.*

Proof: Clearly

$$\{\nabla u = 0\} = \{\nabla u = 0 \quad \wedge \quad f = 0\} \cup \{\nabla u = 0 \quad \wedge \quad f \neq 0\},$$

being this a disjoint union. Moreover, by Theorem 3.1.18,
$|\{\nabla u = 0 \quad \wedge \quad f \neq 0\}| = 0$. Hence

$$|\{\nabla u = 0\}| = |\{\nabla u = 0 \quad \wedge \quad f = 0\}| \leq \{|f = 0\}| = 0$$

obtaining the desired result. □

Corollary 3.1.20. *If u is an eigenfunction corresponding to an arbitrary eigenvalue λ of Dirichlet p- Laplacian, $1 < p < \infty$, then the Lebesgue measure of $\{x \in \Omega : \nabla u(x) = 0\}$ is zero.*

Proof: We note that since u is a continuous function in $\overline{\Omega}$ and $u = 0$ on $\partial\Omega$, we have that u and so u^{p-1} is upper bounded in Ω. Moreover, as Ω is bounded domain, we have λu^{p-1} belongs to $L^{\gamma}(\Omega)$ for any $\gamma > \dfrac{n}{p}, \gamma \geq 2.$. Therefore, we can apply the Corollary 3.1.19 to eigenfunctions of an arbitrary eigenvalue λ, with $f = \lambda u^{p-1}$ which is positive in Ω. □

3.2 Properties of geometric variations

This section is based on [22]. Now we are interested in the continuity of the function

$$\Omega \mapsto \lambda_1(\Omega)$$

whose domain is the class $\mathscr{C}$ of the open subsets of a fixed closed ball D. We consider the topology induced by the Hausdorff distance in $\mathscr{C}$.

Let us define different kinds of convergence which will be useful. Let (Ω_n) be a sequence of open subsets of D. Let $\Omega \subset D$ an open set.

Definition 3.2.1. *We say that the sequence of the spaces $W_0^{1,p}(\Omega_n)$ converges in the* ***sense of Mosco*** *to the space $W_0^{1,p}(\Omega)$ if the following conditions hold:*

1. *For all $\varphi \in W_0^{1,p}(\Omega)$, there exists a sequence $\varphi_n \in W_0^{1,p}(\Omega_n)$ such that φ_n converges strongly in $W_0^{1,p}(D)$ to φ.*

2. *For every sequence $\varphi_n \in W_0^{1,p}(\Omega_n)$ weakly convergent in $W_0^{1,p}(D)$ to a function φ, we have $\varphi \in W_0^{1,p}(\Omega)$.*

Definition 3.2.2. *We say a sequence (Ω_n) γ_p - **converges** to Ω if for every $f \in W^{-1,q}(\Omega)$, the solutions of the Dirichlet problem*

$$-\Delta_p u_n = f \ \textit{in} \ \Omega_n, \ u_n \in W_0^{1,p}(\Omega_n)$$

converge in $W_0^{1,p}(D)$ to the solution of

$$-\Delta_p u = f \ \textit{in} \ \Omega_n \ u \in W_0^{1,p}(\Omega).$$

Let us denote by

$$\vartheta_l(D) = \{\Omega \subset D : \Omega \text{ is open and } \sharp\Omega^c \leq l\}$$

where $\sharp$ is the number of connected components. We have the following theorem.

Theorem 3.2.3. *(Bucur- Trebeschi). Let $n \geq p > n-1$. Consider a sequence $(\Omega_n) \subset \vartheta_l(D)$ and assume that Ω_n converges with respect to the Hausdorff distance to Ω. Then $\Omega \subset \vartheta_l(D)$ and (Ω_n) γ_p- converges to Ω.*

Proof: See [6]. □

Remark 3.2.4. *If $p > n$, a sequence of open sets which converges with respect to Hausdorff distance to an open set, also γ_p converges to such set. This follows directly from the embebbeding of $W_0^{1,p}(D) \subset W^{\varepsilon,\infty}(D)$ and the characterization of $W_0^{1,p}(\Omega)$ (see [6]). Thus, the previous theorem is non trivial just for $n-1 < p \leq n$.*

Proposition 3.2.5. *Let $1 < p < \infty$. Let $\Omega_n \subset \vartheta_l(D)$ be an open set. If (Ω_n) γ_p-converges to Ω, then $W_0^{1,p}(\Omega_n)$ converges in the sense of Mosco to the space $W_0^{1,p}(\Omega)$.*

Proof: For proving the first condition we choose $u \in W_0^{1,p}(\Omega)$ and consider $u_n \in W_0^{1,p}(\Omega_n)$ such that

$$-\Delta_p u_n = -\Delta_p u \text{ in } \Omega_n.$$

As (Ω_n) γ_p- converges to Ω, we have (u_n) converges to u in $W_0^{1,p}(D)$.

For proving the second condition of Mosco, consider a sequence $u_n \in W_0^{1,p}(\Omega_n)$ weakly convergent in $W_0^{1,p}(D)$ to the function u. We need to show that $u \in W_0^{1,p}(\Omega)$.

Let $v_n \in W_0^{1,p}(\Omega_n)$ be such that

$$\Delta_p v_n = -\Delta_p u \text{ in } \Omega_n, \tag{3.2.1}$$

and $v \in W_0^{1,p}(\Omega)$ be such that

$$-\Delta_p v = -\Delta_p u \text{ in } \Omega. \tag{3.2.2}$$

As (Ω_n) γ_p- converges to Ω, we have that v_n converges strongly to v in $W_0^{1,p}(D)$. We will show that $v = u$ which will prove that $u \in W_0^{1,p}(\Omega)$. Taking $v_n - u_n$ as a test function in the weak formulation of (3.2.1), we have

$$\int_{\Omega_n} |\nabla v_n|^{p-2} \langle \nabla v_n, \nabla(v_n - u_n)\rangle = \int_{\Omega_n} (-\Delta_p u)(v_n - u_n) \tag{3.2.3}$$

Since that (u_n) converges weakly to u in $W_0^{1,p}(D)$ and v_n converges to v strongly in in the same space, we can take the limit as n tends to infty in (3.2.3). Therefore, we get

$$\int_D |\nabla v|^{p-2} \langle \nabla v, \nabla(v - u)\rangle = \int_D (-\Delta_p u)(v - u).$$

But as

$$\int_D (-\Delta_p u)(v - u) = \int_D |\nabla v|^{p-2} \langle \nabla v, \nabla(v - u)\rangle,$$

we have

$$\int_D \langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla(v - u)\rangle = 0.$$

By the strict convexity of function $A(\xi) = |\xi|^p$ we have $\nabla(u - v) = 0$. This implies that $u = v$ in D. Since that $v = 0$ in $\partial\Omega$, we conclude that $u = 0$ on $\partial\Omega$. Thus, $u \in W_0^{1,p}(\Omega)$ and the second condition of the Mosco convergence is proved. □

We will use the above results to prove the next theorem, which is a generalization of Sverak's result for the Laplacian (see [14]).

Theorem 3.2.6. *Let $p > n-1$. Consider a sequence $(\Omega_n) \subset \vartheta_l(D)$ and assume that Ω_n converges with respect to the Hausdorff distance to Ω. Then $\lambda_1(\Omega_n)$ converges to $\lambda_1(\Omega)$.*

Proof: By Theorem 3.2.3 and Remark 3.2.4, $(\Omega_n)_{n=1}^{\infty}$ γ_p- converges to Ω. By Proposition 3.2.5, $W_0^{1,p}(\Omega_n)$ converges in the sense of Mosco to the space $W_0^{1,p}(\Omega)$. We have

$$\lambda_1(\Omega_n) = \min_{\phi \in W_0^{1,p}(\Omega_n), \phi \neq 0} \frac{\int_{\Omega_n} |\nabla \phi|^p}{\int_{\Omega_n} |\phi|^p} = \int_{\Omega_n} |\nabla u_n|^p$$

where the minimum is obtained by u_n with $\int_{\Omega_n} |u_n|^p = 1$, and

$$\lambda_1(\Omega) = \min_{\phi \in W_0^{1,p}(\Omega), \phi \neq 0} \frac{\int_{\Omega} |\nabla \phi|^p}{\int_{\Omega} |\phi|^p} = \int_{\Omega} |\nabla u|^p$$

where the minimum is obtained by u with $\int_{\Omega} |u|^p = 1$. Let $d \geq 0$ and let $\Omega' \subset \Omega$ be an open set with $d_H(\Omega', \Omega) \leq d$. As by hypothesis Ω_n converges with respect to the Hausdorff distance to Ω, we have that for n large enough, $\Omega' \subset \Omega_n$ and so

$$\int_D |\nabla u_n|^p = \int_{\Omega_n} |\nabla u_n|^p = \lambda_1(\Omega_n) \leq \lambda_1(\Omega').$$

Thus we conclude that $(u_n)_{n=1}^{\infty}$ is bounded in $W_0^{1,p}(D)$. Assume without loss of generality $(u_n)_{n=1}^{\infty}$ converges weakly in this space to a function $\hat{u}$. The second condition of Mosco implies that $\hat{u} \in W_0^{1,p}(\Omega)$. Also, by Theorem 1.2.8, $(u_n)_{n=1}^{\infty}$ converges strongly to $\hat{u}$ in $L^p(D)$. This gives,

$$\int_{\Omega} |\hat{u}|^p = \int_D |\hat{u}|^p = \lim_{n \to \infty} \int_D |u_n|^p = 1.$$

Using the weak lower semicontinuity of the norm $W_0^{1,p}$ and as $\hat{u} \in W_0^{1,p}(\Omega)$, we get

$$\liminf_{n \to \infty} \int_{\Omega_n} |\nabla u_n|^p \geq \int_D |\nabla \hat{u}|^p = \int_{\Omega} |\nabla \hat{u}|^p$$

Therefore

$$\liminf_{n \to \infty} \lambda_1(\Omega_n) \geq \lambda_1(\Omega).$$

Using the first condition of Mosco, there exists a sequence $(v_n)_{n=1}^{\infty}$ with $v_n \in W_0^{1,p}(\Omega_n)$ for all n, such that v_n converges strongly in $W_0^{1,p}(D)$ to u. Thus

$$\lambda_1(\Omega_n) \leq \frac{\int_{\Omega_n} |\nabla v_n|^p}{\int_{\Omega_n} |v_n|^p},$$

and this implies that

$$\begin{aligned}\limsup \lambda_1(\Omega_n) &\leq \limsup \frac{\int_D |\nabla v_n|^p}{\int_D |v_n|^p} \\ &= \lim_{n\to\infty} \frac{\int_D |\nabla v_n|^p}{\int_D |v_n|^p} \\ &= \int_\Omega |\nabla u|^p.\end{aligned}$$

Thus

$$\limsup_{n\to\infty} \lambda_1(\Omega_n) \leq \lambda_1(\Omega).$$

Therefore

$$\lambda_1(\Omega) \geq \limsup_{n\to\infty} \lambda_1(\Omega_n) \geq \liminf_{n\to\infty} \lambda_1(\Omega_n) \geq \lambda_1(\Omega).$$

We conclude that $\lambda_1(\Omega_n)$ converges to $\lambda_1(\Omega)$. □

3.3 Derivative of the first eigenvalue

This section is based on [11]. Our aim is to show that $\lambda_1(\Omega)$ is differentiable when differentiable perturbations of the domain Ω are considered. So consider a family of diffeomorphisms $T = T_\delta(x) \in C^1(\overline{\Omega}, \mathbb{R}^n)$, with small $\delta > 0$, such that

$$T_\delta(x) = x + \delta R(x) + S(x, \delta), \tag{3.3.1}$$

where $R, S(\cdot, \delta) \in C^1(\overline{\Omega}, \mathbb{R}^n), S(x, \delta) = o(\delta)$ when $\delta \to 0$ in $C^1(\overline{\Omega}, \mathbb{R}^n)$.

We denote $\Omega_\delta = T_\delta(\Omega)$ and $\lambda(\delta) = \lambda_1(\Omega_\delta)$. By definition of $\lambda(\delta)$ we have,

$$\lambda(\delta) = \inf_{u \in W_0^{1,p}(\Omega_\delta)} \frac{\int_{\Omega_\delta} |\nabla v(y)|^p \, dy}{\int_{\Omega_\delta} |v(y)|^p \, dy}. \tag{3.3.2}$$

We consider $x = T_\delta^{-1}(y)$ in Ω, i.e. $y \approx x + \delta R(x)$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. If $v(y) = u(x)$ then by chain rule

$$\frac{\partial v}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \cdot \frac{\partial x_j}{\partial y_i} = \left(DT_\delta^{-1}\right)^* \nabla u. \tag{3.3.3}$$

In this way,

$$\lambda(\delta) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |D(x,\delta)\nabla u|^p C(x,\delta)\; dx}{\int_\Omega |u|^p C(x,\delta)\; dx},$$

where $D(x,\delta) = \left(DT_\delta^{-1}\right)^*, C(x,\delta) = |det(DT_\delta(x))|$. The matrix $\left(\frac{\partial x_j}{\partial y_i}\right)^*$ is such that

$$\left(\frac{\partial x_j}{\partial y_i}\right)^* = \left((DT_\delta)^{-1}\right)^* \approx \left((I + \delta DR)^{-1}\right)^* \approx \left(I - \delta(DR)^*\right), \tag{3.3.4}$$

where the neglected terms involve δ^2 or higher powers of δ.

The next theorem is a continuity result.

Theorem 3.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$ boundary, $\lambda(\delta)$ the first eigenvalue of p-Laplacian in the domain $\Omega_\delta = T_\delta(\Omega)$, where T_δ is a family of C^1 diffeomorphisms verifying (3.3.1). Then the function $\lambda = \lambda(\delta)$ is continuous at $\delta = 0$, and if ϕ_δ denotes the positive eigenfunction in Ω_δ normalized as $|\phi_\delta|_\infty = 1$ we have*

$$\phi_\delta \to \phi$$

in $C_0^{1,\beta}$, as $\delta \to 0$ for some $0 < \beta < 1$, where ϕ is the positive normalized first eigenfunction for $\delta = 0$.

Proof: See GarcÃŋa et.al.[11]. □

Theorem 3.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$ boundary, $\Omega_\delta = T_\delta(\Omega)$ the perturbation of Ω asociated to a family $T_\delta(x) = x + \delta R(x) + S(x,\delta)$. Then the first eigenvalue $\lambda(\delta) = \lambda_1(\Omega_\delta)$ of $-\Delta_p$ is differentiable with respect to δ at $\delta = 0$. Moreover*

$$\lambda'(0) = -(p-1) \int_{\partial\Omega} \langle R, n \rangle \left|\frac{\partial \phi}{\partial n}\right|^p \; d\sigma, \tag{3.3.5}$$

where n is the outward unit normal vector and ϕ the positive first eigenfunction in Ω normalized so that

$$\int_\Omega |\phi|^p \ dx = 1.$$

Proof: We recall that

$$\lambda(\delta) = \inf_{u \in W_0^{1,p}(G)} \frac{\int_\Omega |D(x,\delta)\nabla u|^p C(x,\delta) \ dx}{\int_\Omega |u|^p C(x,\delta) \ dx},$$

where $D(x,\delta) \approx I - \delta(DR)^*, C(x,\delta) = |det(DT_\delta(x))|$. Thus, we may take

$$A(x,\delta,\xi) = (|D(x,\delta)\cdot\xi|^2)^{\frac{p}{2}} C(x,\delta)) \text{ for all } \xi \in \mathbb{R}^n$$

and

$$B(x,\delta,z) = |z|^p C(x,\delta) \text{ for all } z \in \mathbb{R}$$

in Theorem 2.3.1. Now we calculate A_1 and B_1.

$$\begin{aligned} A_1(x,\delta,\xi) &= |\xi|^p \partial_\delta C(x,0) + (\partial_\delta(\langle D(x,\delta)\cdot\xi, D(x,\delta)\cdot\xi\rangle)^{\frac{p}{2}})(x,0,\xi) \\ &= |\xi|^p \text{div} R(x) + \frac{p}{2}|\xi|^{p-2}(\langle D_\delta(x,0)\xi,\xi\rangle + \langle\xi, D_\delta(x,0)\xi\rangle) \\ &= |\xi|^p \text{div} R(x) + \frac{p}{2}|\xi|^{p-2}(\langle -(DR)^*\xi,\xi\rangle + \langle\xi, -(DR)^*\xi\rangle) \\ &= |\xi|^p \text{div} R(x) - p|\xi|^{p-2}\langle\xi, DR\xi\rangle \end{aligned} \tag{3.3.6}$$

and

$$B_1(x,0,z) = |z|^p \partial_\delta C(x,0) = |z|^p \text{div} R(x). \tag{3.3.7}$$

Therefore, by (3.3.6), (3.3.7), (2.3.2) and the fact that φ is normalized, we havee

$$\lambda'(0) = \int_\Omega \left(|\nabla\phi|^p \text{div} R(x) - p|\nabla\phi|^{p-2}\langle\nabla\phi, DR(x)\nabla\phi\rangle\right) \ dx - \lambda_0 \int_\Omega |\phi|^p \text{div} R(x) \ dx, \tag{3.3.8}$$

where λ_0 is the first eigenvalue for the domain Ω. Our intention is to perform an integration by parts in this expression. The lack of C^2 regularity of ϕ leads us to consider the problem

$$\begin{cases} -\text{div}((\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u) = \lambda_0 \ \phi^{p-1} & \text{in}\Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3.9}$$

for $\varepsilon > 0$ small. As a consequence of the quasilinear theory in [13], this problem has a unique solution $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ and since $\lambda_0\phi^{p-1} \geq 0$, it follows that $u_\varepsilon \geq 0$. Further, the uniform $C^{1,\beta}$ estimates in $\varepsilon > 0$ implies that there exists $u \in C^1(\overline{\Omega})$ such that $u_\varepsilon \to u$ in $C^1(\overline{\Omega})$ as $\varepsilon \to 0$. Moreover since for all $\psi \in D(\Omega)$,

$$\int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, \nabla \psi\rangle \, dx = \lambda_0 \int_\Omega \phi^{p-1}\psi \, dx,$$

by taking limit as $\varepsilon \to 0$,

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \psi\rangle \, dx = \lambda_0 \int_\Omega \phi^{p-1}\psi \, dx.$$

Thus, both u and ϕ are weak solutions of the problem $-\Delta_p v = \lambda_0\phi^{p-1}$ with the Dirichlet condition. By uniqueness of solution, it follows that $u = \phi$. Therefore $u_\varepsilon \to \phi$ in $C^1(\overline{\Omega})$ as $\varepsilon \to 0$.

Now, considering $\langle R, \nabla u_\varepsilon\rangle$ as a test function in (3.3.9) and observing that this function does not vanishes on $\partial\Omega$, we have

$$\begin{aligned}\int_\Omega \lambda_0\phi^{p-1}\langle R, \nabla u_\varepsilon\rangle \, dx = &\int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, \nabla(\langle R, \nabla u_\varepsilon\rangle)\rangle \, dx \\ &- \int_{\partial\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, n\rangle\langle R, \nabla u_\varepsilon\rangle \, d\sigma.\end{aligned} \tag{3.3.10}$$

Also we get that

$$\begin{aligned}\int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, \nabla(\langle R, \nabla u_\varepsilon\rangle)\rangle \, dx &= \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, D^2u_\varepsilon R + DR\cdot\nabla u_\varepsilon\rangle \, dx \\ &= \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle D^2u_\varepsilon \nabla u_\varepsilon, R\rangle \, dx \\ &+ \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, DR\cdot\nabla u_\varepsilon\rangle \, dx \\ &= \int_\Omega \left\langle \frac{1}{p}\nabla\left((\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p}{2}}\right), R\right\rangle \, dx \\ &+ \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, DR\cdot\nabla u_\varepsilon\rangle \, dx\end{aligned} \tag{3.3.11}$$

Replacing (3.3.10) in (3.3.11) we obtain

$$\begin{aligned}\int_\Omega \lambda_0 \phi^{p-1}\langle R, \nabla u_\varepsilon\rangle \, dx = &\int_\Omega \left\langle \frac{1}{p}\nabla\left(\left(\varepsilon+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\right), R\right\rangle \, dx \\ &+ \int_\Omega \left(\varepsilon+|\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, DR\cdot\nabla u_\varepsilon\rangle \, dx \\ &- \int_{\partial\Omega} \left(\varepsilon+|\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, n\rangle\langle R, \nabla u_\varepsilon\rangle \, d\sigma.\end{aligned} \tag{3.3.12}$$

Letting ε tend to 0 in (3.3.12) we have

$$\begin{aligned}\int_\Omega \lambda_0 \phi^{p-1}\langle R, \nabla \phi\rangle \, dx = &\int_\Omega \left\langle \frac{1}{p}\nabla\left(|\nabla\phi|^p\right), R\right\rangle \, dx \\ &+ \int_\Omega |\nabla\phi|^{p-2}\langle \nabla \phi, DR\cdot\nabla \phi\rangle \, dx \\ &- \int_{\partial\Omega} |\nabla\phi|^{p-2}\langle \nabla \phi, n\rangle\langle R, \nabla \phi\rangle \, d\sigma.\end{aligned} \tag{3.3.13}$$

By integration by parts and considering that $\phi = 0$ on $\partial\Omega$, we obtain

$$\int_\Omega \phi^{p-1}\langle R, \nabla\phi\rangle \, dx = \int_\Omega \left\langle R, \nabla\left(\frac{\phi^p}{p}\right)\right\rangle \, dx = -\int_\Omega \frac{\phi^p}{p}\operatorname{div}R \, dx. \tag{3.3.14}$$

Also, using integration by parts we have

$$\int_\Omega \left\langle \frac{1}{p}\nabla\left(|\nabla\phi|^p\right), R\right\rangle \, dx = \int_{\partial\Omega}\left\langle \frac{|\nabla\phi|^p}{p}R, n\right\rangle \, d\sigma - \int_\Omega \frac{|\nabla\phi|^p}{p}\operatorname{div}R \, dx. \tag{3.3.15}$$

So that by (3.3.14) and (3.3.15), (3.3.13) gives

$$\begin{aligned}-\int_\Omega \lambda_0 \frac{\phi^p}{p} \operatorname{div}R \, dx = &\int_{\partial\Omega}\left\langle \frac{|\nabla\phi|^p}{p}R, n\right\rangle \, d\sigma - \int_\Omega \frac{|\nabla\phi|^p}{p}\operatorname{div}R \, dx \\ &+ \int_\Omega |\nabla\phi|^{p-2}\langle \nabla \phi, DR\cdot\nabla \phi\rangle \, dx - \int_{\partial\Omega} |\nabla\phi|^{p-2}\langle \nabla \phi, n\rangle\langle R, \nabla \phi\rangle \, d\sigma.\end{aligned} \tag{3.3.16}$$

Since $\phi = 0$ on $\partial\Omega, \nabla\phi = \dfrac{\partial\phi}{\partial n}n$ in $\partial\Omega$ and n is the outward unit normal vector, (3.3.16) now becomes

$$\begin{aligned}-\int_\Omega \lambda_0 \frac{\phi^p}{p} \operatorname{div}R \, dx = &\int_{\partial\Omega}\left\langle \frac{1}{p}\left|\frac{\partial\phi}{\partial n}\right|^p R, n\right\rangle \, d\sigma - \int_\Omega \frac{|\nabla\phi|^p}{p}\operatorname{div}R \, dx \\ &+ \int_\Omega |\nabla\phi|^{p-2}\langle \nabla \phi, DR\cdot\nabla \phi\rangle \, dx - \int_{\partial\Omega} \left|\frac{\partial\phi}{\partial n}\right|^p \langle R, n\rangle \, d\sigma.\end{aligned} \tag{3.3.17}$$

Therefore, using (3.3.17) in (3.3.8), we get

$$\lambda'(0) = -(p-1)\int_{\partial\Omega} \langle R, n\rangle \left|\frac{\partial\phi}{\partial n}\right|^p d\sigma. \tag{3.3.18}$$

□

Chapter 4

Minimization of the first eigenvalue for the Dirichlet p- Laplacian

4.1 Existence and uniqueness of a minimizer for the first eigenvalue among domains with the same volume

There is a large history of eigenvalue minimization problems, especially the Dirichlet Laplacian problem. A famous conjecture made by Lord Rayleigh is the following: "The disk should minimize the first eigenvalue of the Laplacian Dirichlet among every open set of given measure". This conjecture was proved simultaneously and independently by G. Faber and E. Krahn. We shall deal with the $p-$ Laplacian version of this theorem.

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let B be a ball of the same volume as Ω, then*

$$\lambda_1(B) \leq \lambda_1(\Omega).$$

Proof: Let Ω be a domain and let Ω^* be its Schwarz symmetrization. Recall that, by

Proposition 3.1.7, λ_1 is invariant by translation. Thus, if B is a ball with the same volume as Ω, then $\lambda_1(B) = \lambda_1(\Omega^*)$. In this way, it is sufficient to work with Ω^* instead of an arbitrary ball B.

Let u be an eigenfunction asociated to $\lambda_1(\Omega)$ and u^* its Schwarz symmetrization. Taking the function $F(x) = |x|^p$ in (2.1.9), we have that

$$\int_\Omega |u|^p = \int_{\Omega^*} |u^*|^p. \tag{4.1.1}$$

Further, the PÃşlya-SzegÃű inequality implies that

$$\int_{\Omega^*} |\nabla u^*|^p dx \leq \int_\Omega |\nabla u|^p dx. \tag{4.1.2}$$

By (4.1.1), (4.1.2) and since u is an eigenfunction of $\lambda_1(\Omega)$ we get

$$\frac{\int_{\Omega^*} |\nabla u^*|^p}{\int_{\Omega^*} |u^*|^p} \leq \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} = \lambda_1(\Omega).$$

This implies

$$\lambda_1(\Omega^*) = \min_{\varphi \in W_0^{1,p}(\Omega^*), \varphi \neq 0} \frac{\int_{\Omega^*} |\nabla \varphi|^p}{\int_{\Omega^*} |\varphi|^p} \leq \frac{\int_{\Omega^*} |\nabla u|^p}{\int_{\Omega^*} |u|^p} \leq \lambda_1(\Omega).$$

□

We now show the uniqueness of the minimizer for λ_1 among domains with the same volume. This result has been shown by Bhatacharya.T.[4]. This proof is pretty technical and is based on Talenti's Theorem. We give two proofs. The first one is based on a classical result of Brothers and Ziemer [5] and this should be found in Alvino et. al. [1]. Later, we give an original proof by considering an overdetermined problem á la Serrin [24] for domains with C^2 boundary.

Let us first consider the proof based the result of Brothers and Ziemer.

Theorem 4.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let B be a ball with the same volume as Ω. If Ω is not a ball then*

$$\lambda_1(\Omega) > \lambda_1(B).$$

Proof: Let $\Omega \subset \mathbb{R}^n$ be a domain. By the observations made in the beginning of the proof of Theorem 4.1.1 is sufficient to work with Ω^* instead of an arbitrary ball B.

Let u be an eigenfunction asociated to $\lambda_1(\Omega)$ and u^* its Schwarz symmetrization. If $\lambda_1(\Omega) = \lambda_1(\Omega^*)$, then

$$\lambda_1(\Omega^*) = \lambda_1(\Omega) = \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \geq \frac{\int_{\Omega^*} |\nabla u^*|^p}{\int_{\Omega^*} |u^*|^p} \geq \lambda_1(\Omega^*).$$

Thus,

$$\int_\Omega |\nabla u|^p = \int_B |\nabla u^*|^p.$$

It also follows that the Rayleigh quotient of u^* is equal to $\lambda_1(\Omega^*)$, and therefore u^* is an eigenfunction corresponding to $\lambda_1(\Omega^*)$. Thus, by Corollary 3.1.20 we have that $|C^*| = 0$, where $C^* = \{\nabla u^* = 0\}$. It implies that $|C^* \cap {u^*}^{-1}(0, M)| = 0$. So, the hypotesis of the second part of the Corollary 2.1.16 is satisfied . Therefore, we conclude that there exists a translation of u^* which is equal almost everywhere to u. However, as u, u^* are C^1 functions in their respective domains (Proposition 3.1.11), we have indeed that the translation of u^* is equal to u everywhere in $\overline{\Omega}$. Therefore Ω is a ball. $\square$

Next, we give the new proof, for domains with C^2 boundary

Theorem 4.1.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain with C^2 boundary which minimizes λ_1 among domains of given volume. Then, there exists a negative constant c such that the eigenfunction u, satisfy*

$$\frac{\partial u}{\partial n} = c \text{ on } \partial\Omega.$$

Proof: We observe that if Ω minimizes λ_1 under the condition that $Vol(\Omega) = A$, then there exists a Lagrange multiplier C such that $\lambda_1'(0) = C \cdot Vol'(0)$. So, it follows by Theorems 3.3.2 and 2.4.1 that

$$-(p-1)\int_{\partial\Omega} \langle R, n\rangle \left|\frac{\partial u}{\partial n}\right|^p d\sigma = C\int_{\partial\Omega} \langle R, n\rangle \, d\sigma,$$

for every any vector field $R(\cdot) \in C^1(\overline{\Omega})$. But this implies that

$$\int_{\partial\Omega} \left[-(p-1)\left|\frac{\partial u}{\partial n}\right|^p - C\right] \langle R, n\rangle \, d\sigma = 0. \tag{4.1.3}$$

As R is arbitrary, (4.1.3) implies that

$$-(p-1)\left|\frac{\partial u}{\partial n}\right|^p = C.$$

Thus

$$\left|\frac{\partial u}{\partial n}\right| = \left(\frac{-C}{p-1}\right)^{\frac{1}{p}}, \text{ in } \partial\Omega.$$

But by Proposition 3.1.14, $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$.

Therefore, we conclude that

$$\frac{\partial u}{\partial n} = c \text{ on } \partial\Omega.$$

with $c = -\left(\frac{-C}{p-1}\right)^{\frac{1}{p}}$. □

Consider Ω a bounded domain with C^2 boundary in $\mathbb{R}^n$. Suppose there exists a positive function u satisfying

$$\begin{cases} -\Delta_p u = \lambda_1 u^{p-1} & \text{in}\Omega \\ u = 0 & \text{in } \partial\Omega \\ \frac{\partial u}{\partial n} = constant & \text{in } \partial\Omega \end{cases} \tag{4.1.4}$$

in the weak sense. Must be Ω a ball?. In the afirmative case, we will have shown that the ball is the unique minimizer of λ_1 among bounded domains with C^2 boundary such that they have the same volume. Indeed, this result is proved in the next theorem.

Theorem 4.1.4. *Let Ω be a bounded domain with C^2 boundary in $\mathbb{R}^n$. Suppose that there exists a positive function u satisfying (4.1.4) in the weak sense. Then Ω is a ball.*

Proof: Let T_0 be a hyperplane in $\mathbb{R}^n$ not intersecting the domain Ω. We suppose this plane to be moved continously parallel to T_0 to new positions, until ultimately it begins to intersect Ω. From that moment onward, at each stage the resulting plane T cuts off from Ω a cap $\Sigma(T)$ which corresponds to the portion of Ω that lies on the same side of T as the original plane T_0.

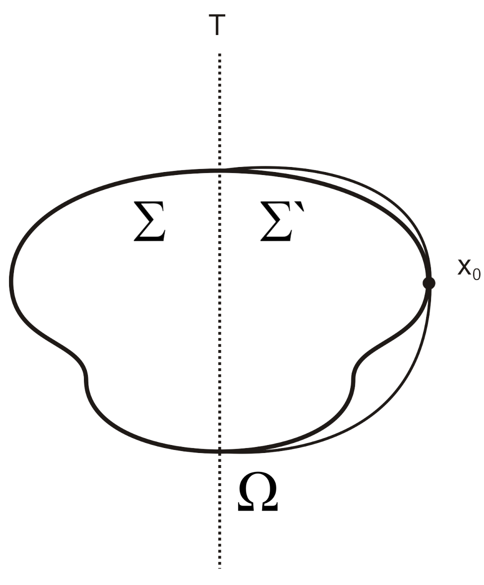
For any cap $\Sigma(T)$ thus formed, we let $\Sigma'(T)$ be its reflection in T. We note that $\Sigma'(T)$ is contained in Ω until $\Sigma'(T)$ becomes internally tangent to the boundary of Ω

at some point P not on T. Denote the plane T when it reaches this position by T_α and $\Sigma(T_\alpha)$ by Σ'. Our aim is to prove that Ω is symmetric with respect to T_α, i.e., $\Sigma' = \Omega \cap T_\alpha^+$, where T_α^+ is the half plane Ω formed by T_α not containing T_0. If we suceed in proving this, then we may conclude that for any given direction in $\mathbb{R}^n$, we can find a plane T_α about which Ω is symmetric. But the only domain which have this symmetry property are balls. Thus, our theorem would be proved.

Since, $\Sigma' \subset \Omega \cap T_\alpha^+$, by Proposition 1.5.1 it is enough to show that $\partial\Sigma' \subset \partial(\Omega \cap T_\alpha^+)$, in order to conclude $\Sigma' = \Omega \cap T_\alpha^+$. But as $\partial\Sigma' \subset (\partial\Sigma' \cap T_\alpha^+) \cup (\Omega \cap T_\alpha)$ and $\partial(\Omega \cap T_\alpha^+) \subset (\partial\Omega \cap T_\alpha^+) \cup (\Omega \cap T_\alpha)$. Thus, it is sufficient to show that

$$\partial\Sigma' \cap T_\alpha^+ \subset \partial\Omega \cap T_\alpha^+.$$

Assuming that $\partial\Sigma' \cap T_\alpha^+ \not\subseteq \partial\Omega \cap T_\alpha^+$, we show that this leads to a contradiction. Consider a point $x_0 \in \partial\Sigma' \cap T_\alpha^+$ which belongs to $\partial\Omega \cap T_\alpha^+$ where Σ' is internally tangent to $\partial\Omega \cap T_\alpha^+$ and such that every neighborhoord of x_0 contains points of $\partial\Sigma' \cap T_\alpha^+$ which does not belong to $\partial\Omega \cap T_\alpha^+$.



For proving the assertion, we introduce a new function v defined in Σ' by $v(x) =$

$u(x')$, where x' is the reflected value of x across T_α. We note that u is such that:

$$-\Delta_p u = \lambda_1 u^{p-1} \text{ in } \Sigma',$$
$$u \geq 0 \text{ in } \partial\Sigma',$$
$$\frac{\partial u}{\partial n} = c \text{ in } \partial\Sigma' \cap \partial\Omega;$$

while v satisfies

$$-\Delta_p v = \lambda_1 v^{p-1} \text{ in } \Sigma',$$
$$v = 0 \text{ in } \partial\Sigma' \setminus T_\alpha,$$
$$v = u \text{ in } \partial\Sigma' \cap T_\alpha,$$
$$\frac{\partial v}{\partial n} = c \text{ in } \partial\Sigma' \setminus T_\alpha.$$

in the weak sense.

Step 1: We note that

$$-\text{div}(A(\nabla u) - A(\nabla v)) = -\Delta_p u + \Delta_p v = \lambda_1(u^{p-1} - v^{p-1}) \geq 0 \text{ in } N \tag{4.1.5}$$

in the weak sense, where $A(\xi) = |\xi|^{p-2} \cdot \xi$ for $\xi = (\xi_1, \dots \xi_n) \in \mathbb{R}^n$. Let $w = u - v$. By the mean value theorem,

$$\begin{aligned}
A(\nabla u) - A(\nabla v) &= A(t\nabla u + (1-t)\nabla v)|_0^1 \\
&= \int_0^1 \frac{d}{dt} A(t\nabla u + (1-t)\nabla v)\ dt \\
&= \int_0^1 \left(\left\langle \int_0^1 (\nabla A_i)(t\nabla u(x) + (1-t)\nabla v(x))dt, \nabla w \right\rangle\right)_{i=1}^n\ dt \\
&= \left(\left\langle \int_0^1 (\nabla A_i)(t\nabla u(x) + (1-t)\nabla v(x))dt, \nabla w \right\rangle\ dt\right)_{i=1}^n.
\end{aligned}$$

Thus by (4.1.5)

$$-\sum_{i=1}^n \left(\left\langle \int_0^1 (\nabla A_i)(t\nabla u(x) + (1-t)\nabla v(x))dt, \nabla w \right\rangle\ dt\right)_{i=1}^n \geq 0$$

in the weak sense. Let $a_{ij}(x) = \displaystyle\int_0^1 \frac{\partial A_i}{\partial x_j}(t\nabla u(x) + (1-t)\nabla v(x))dt$, the function w satisfies

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) \geq 0 \text{ in } \Sigma',$$

$$w \geq 0 \text{ in } \partial Sigma' \setminus T_\alpha,$$

$$w = 0 \text{ in } \partial\Sigma' \cap T_\alpha,$$

$$w = 0 \text{ in } \partial\Sigma' \cap (\partial\Omega \cap T_\alpha^+).$$

Step 2. We shall now that $((a_{ij}))$ is uniformly positive definite in a neighborhood N of x_0. By Proposition 1.5.3, there exists K such that

$$\langle (a_{ij})(t\nabla u(x) + (1-t)\nabla v(x))\xi, \xi\rangle \geq K \int_0^1 |t\nabla u(x) + (1-t)\nabla v(x)| \ dt\ |\xi|^2. \quad (4.1.6)$$

At x_0, we can write

$$\nabla v(x_0) = \frac{\partial v}{\partial n}(x_0)n + \nabla_{\partial\Sigma' \cap T_\alpha^+} v(x_0)$$

where $\nabla_{\partial\Sigma' \cap T_\alpha^+} v$ is the tangential component in x_0. Since $v = 0$ in $\partial\Sigma' \cap T_\alpha^+$, we have $\nabla_{\partial\Sigma' \cap T_\alpha^+} v(x_0) = 0$, thus

$$\nabla v(x_0) = \frac{\partial v}{\partial n}(x_0)n.$$

Let us define

$$g(t,x) = |t\nabla u(x) + (1-t)\nabla v(x)|^{p-2}.$$

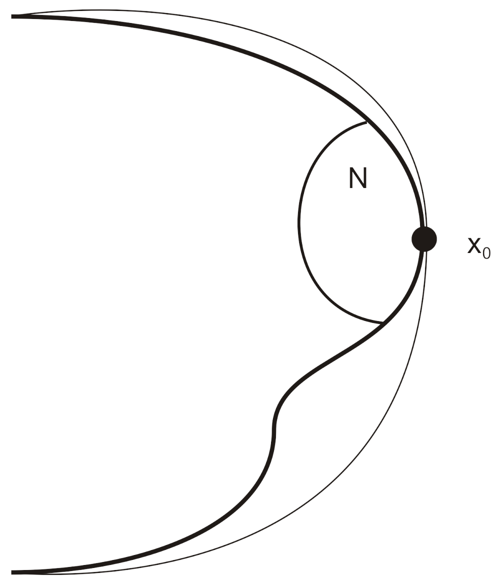
We observe that $g(t,x)$ is continuous with respect to t and x and as the normal derivative of v in x_0 is such that $\dfrac{\partial v}{\partial n} < 0$, we have $g(0,x_0) = \left|\dfrac{\partial v}{\partial n}(x_0)\right| = |c| > 0$. Thus, we can find a neighborhood $[0,t_0] \times N$ where N is a neighborhood of x_0 in Σ' and $t_0 \leq 1$, such that

$$g(t,x) \geq \delta, \ \forall (t,x) \in [0,t_0] \times N$$

for some positive δ.

Therefore ,

$$\int_0^{t_0} g(t,x)\ dt\ |\xi|^2 \geq \delta t_0 |\xi|^2.$$



Thus, the operator $L = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$ is elliptic in N and by Theorem 8.19 in [13] we have

$$w = 0 \text{ or } w > 0 \text{ in } N. \tag{4.1.7}$$

But w is not constant in Σ' since Σ' is internally tangent to $\Omega \cap T_\alpha^+$ in x_0 and we have assumed that every neighborhood of x_0 contains points of $\partial\Omega \cap T_\alpha^+ \setminus (\partial\Sigma' \cap T_\alpha^+)$, there exists nearby points to x_0 in $\partial N \cap \partial\Sigma'$ where $w > 0$. It follows that $w > 0$ in N. Then, we can argue like in Theorem 3.1.14 to show that $\frac{\partial w}{\partial n} < 0$ on $\partial N \cap \partial\Sigma'$. This implies that

$$\frac{\partial u}{\partial n}(x_0) < \frac{\partial v}{\partial n}(x_0).$$

It is a contradiction, since $\frac{\partial u}{\partial n}(x_0) = \frac{\partial v}{\partial n}(x_0) = c$. □

The following corollary says the ball is the unique minimizer for λ_1 among domains with the same surface area. The Laplacian version of this corollary for domains on $\mathbb{R}^2$ was first proved by the German mathematician Richard Courant, without the Faber-Krahn inequality. Let us denote the surface area of a domain Ω, by $S(\Omega)$.

Corollary 4.1.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let B be a ball with the same surface area as Ω, then*

$$\lambda_1(\Omega) \geq \lambda_1(B).$$

Moreover, if Ω is not a ball then

$$\lambda_1(\Omega) > \lambda_1(B).$$

Proof: Let $\Omega \subset \mathbb{R}^n$ be a domain and B' a ball with the same volume as Ω. By Theorems 4.1.1 and 4.1.2,

$$\lambda_1(\Omega) \geq \lambda_1(B'),$$

where the equality holds if, and only if Ω is a translate of B'. Further, by the isoperimetric inequality, and since Ω and B' have the same volume, we have

$$S(\Omega) \geq S(B').$$

Furthermore the equality holds if and only if Ω is a translate of B'. Applying a homothety to B' with $k \geq 1$, obtain a ball $B = kB'$ such that $S(B) = S(\Omega)$. Then, by Proposition 3.1.9, we have

$$\lambda_1(B) = \frac{\lambda_1(B')}{k^p}.$$

It follows that

$$\lambda_1(\Omega) \geq \lambda_1(B') \geq \frac{\lambda_1(B')}{k^p} = \lambda_1(B),$$

and the equality holds if and only if Ω is a translate of B' and $k = 1$, that is, if and only if Ω is a translate of B. □

4.2 The case of polygons

Since in the case of arbitrary domains with given volume λ_1 is minimized by the ball and only by the ball, it is natural to think that for planar polygons with given area, the minimum of λ_1 is obtained by the regular polygon. We prove this for the case of triangles.

Theorem 4.2.1. *Let T be an arbitrary triangle. Let T' be an equilateral triangle with the same area as T, then*

$$\lambda_1(T) \geq \lambda_1(T').$$

Proof: The proof uses Steiner symmetrization. Initially we observe that if T is a triangle whose first eigenfunction is u and T^s is the Steiner symmetrization of T, then by Proposition 2.2.10 and the Corollary 2.2.18, proceeding as in Theorem 4.1.1, we obtain

$$\lambda_1(T) \geq \lambda_1(T^s).$$

Let T be an arbitrary triangle. Considering $T_0 = T$, we apply to T a sequence of Steiner symmetrizations with respect to the perpendicular bisector of each side, getting a sequence of triangles which converges with respect to the Hausdorff distance to an equilateral triangle T', which by Proposition 2.2.3, has the same area as T. More precisely, let $n \geq 1$ and let T_n the isosceles triangle obtained in step n. We denote by h_n its height associated to its basis, a_n the length of its basis and A_n the angle subtended by its base with one of the sides of T_n. Consider the Figure 4.2. Since the triangle T_n is isosceles, if we consider the right triangle of angle base $\pi/2 - A_n$, with adjacent side h_n and hypotenuse a_{n+1} we have

$$\sin A_n = \frac{h_n}{a_{n+1}}. \tag{4.2.1}$$

Moreover, as T_n and T_{n+1} have the same area

$$\frac{h_n}{a_{n+1}} = \frac{h_{n+1}}{a_n} = \sin A_n. \tag{4.2.2}$$

Denote by $x_n := \dfrac{h_n}{a_n}$. The relation (4.2.2) gives

$$x_{n+1} \cdot x_n = \frac{h_{n+1}}{a_{n+1}} \cdot \frac{h_n}{a_n} = \sin^2 A_n, \tag{4.2.3}$$

and if anew consider the right triangle of acute angle A_n, with adjacent side h_n, hypotenuse a_{n+1} and base $\dfrac{1}{2}a_n$ we have

$$\tan A_n = \frac{2h_n}{a_n}. \tag{4.2.4}$$

Thus, from (4.2.3) and (4.2.4) we obtain

$$x_{n+1} = \frac{\sin^2 A_n}{x_n} = \frac{\sin^2(\arctan 2x_n)}{x_n} = \frac{4x_n}{1+4x_n^2}.$$

Observe that the sequence defined by $x_{n+1} = \dfrac{4x_n}{1+4x_n^2}$ converges to the fixed point of the function $f(x) = \dfrac{4x}{1+4x^2}$, which is $x = \dfrac{\sqrt{3}}{2}$. In this way, as

$$\tan A_n = 2x_n,$$

if n tends to $+\infty$, then $\tan A_n$ tends to $\sqrt{3}$, that is

$$\lim_{n\to\infty} A_n = \frac{\pi}{3} \tag{4.2.5}$$

which is the measure of angle of an equilateral triangle. We can assume for every $n \geq 0$, T_n is contained in a fixed ball B. In this way, by Theorem 1.3.3, there exists a subsequence still denoted by (T_n) which converges to Hausdorff distance to an open set $T' \subset B$. Further, as the sequence of vertex P_n^j of T_n, with $j = 1, 2, 3$ stay in B, we can also asume that there exists a subsequence still denoted by (P_n^j) which converges to some point P^j in B. Thus, T' is a triangle of vertices $P^j, j = 1, 2, 3$. As the angle A_n can be calculated as function of vertices ($A = \arccos(\langle u, v\rangle |u||v|)$) and the vertices of T_n converge to the vertex of T', then A_n converges to an angle of T' and by (4.2.5), T' is an equilateral triangle.

By Theorem 3.2.3 and Remark 3.2.4, $\lambda_1(T_n)$ converges to $\lambda_1(T')$ for $1 < p < \infty$. Therefore

$$\lambda_1(T') = \lim_{n\to\infty} \lambda_1(T_n) \leq \lambda_1(T).$$

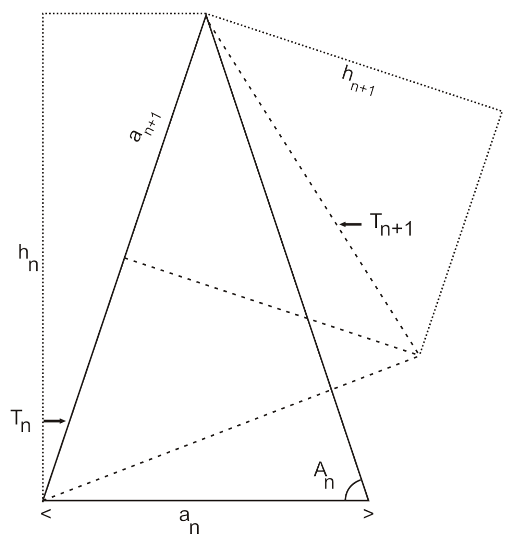


Figure 4.1: The triangle T_n and its Steiner symmetrization T_{n+1}.

□

We show that the equilateral triangle is the unique minimum for the above problem.

Theorem 4.2.2. *Let T be an arbitrary triangle. Let T' be an equilateral with the same area as T. If T is not equilateral then*

$$\lambda_1(T) > \lambda_1(T').$$

Proof: Let T be any triangle which is not equilateral. Then, there exists a side of T for which this triangle is not symmetric with respect to the perpendicular bisector m of such side. Applying a Steiner symmetrization to T with respect m, we get a triangle T^s, which is not congruent with T, such that $\lambda_1(T) \geq \lambda_1(T^s)$. If $\lambda_1(T) = \lambda_1(T^s)$ and u

is an eigenfunction of $\lambda_1(T)$, then

$$\lambda_1(T^s) = \lambda_1(T) = \frac{\int_T |\nabla u|^p}{\int_T |u|^p} \geq \frac{\int_{T^s} |\nabla u^s|^p}{\int_{T^s} |u^s|^p} \geq \lambda_1(T^s).$$

Clearly

$$\int_T |\nabla u|^p = \int_{T^s} |\nabla u^s|^p,$$

and by Theorem 2.2.16 we conclude that T is a translation of T^s, which is a contradiction. Thus, $\lambda_1(T) > \lambda_1(T^s) \geq \lambda_1(T')$. □

Corollary 4.2.3. *Let T be an arbitrary triangle. Let T' be an equilateral triangle with the same perimeter as T, then*

$$\lambda_1(T) \geq \lambda_1(T')$$

Moreover, if T is not an equilateral triangle

$$\lambda_1(T) > \lambda_1(T')$$

Proof: Let T be a arbitrary triangle and T'' the equilateral triangle with the same area as T. By the above theorem

$$\lambda_1(T) \geq \lambda_1(T'')$$

where the equality holds if and only if $T = T''$. Further, by the isoperimetric inequality

$$P(T) \geq P(T'')$$

where the equality holds if and only if $T = T''$. Applying a homothety to T'' with $k \geq 1$, we obtain an equilateral triangle $T' = kT''$ which is such that

$$P(T) = P(T').$$

By Proposition 3.1.10,

$$\lambda_1(T') = \frac{\lambda_1(T'')}{k^p}.$$

It follows that

$$\lambda_1(T) \geq \lambda_1(T'') \geq \frac{\lambda_1(T'')}{k^p} = \lambda_1(T')$$

and the equality holds if and only if $T = T'$. □

We consider the case of quadrilaterals. Applying a sequence of at most 3 Steiner symmetrizations to a given quadrilateral, we transform this to a rectangle. Moreover, as the Steiner symmetrization decreases λ_1 and keeps fixed the area of a quadrilateral, we have the next result.

Theorem 4.2.4. *Let C be any quadrilateral. There exists a rectangle $\hat{C}$ of the same area as C and such that*

$$\lambda_1(C) \geq \lambda_1(\hat{C}) \tag{4.2.6}$$

Proof: We look the most general case. If C is a quadrilateral, then symmetrize C with respect to the straight line orthogonal any diagonal, obtaining a kite $C^{'}$. Next, we symmetrize $C^{'}$ with respect to the perpendicular bisector of a diagonal, getting a rhombus $C^{''}$(See figure 4.2). Finally, if symmetrize $C^{''}$ with respect to the perpendicular bisector of a side, we obtain a rectangle $\hat{C}$, which has equal area as C and (4.2.6) holds.

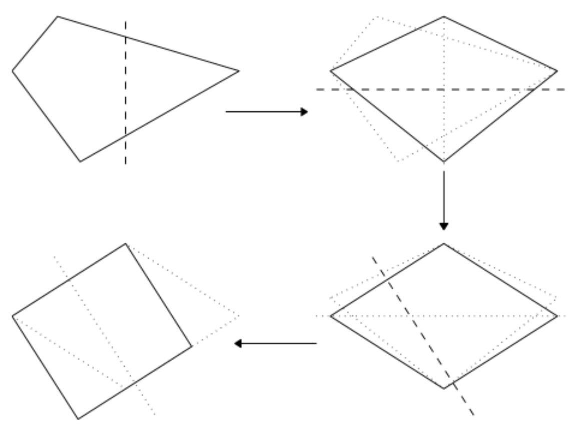


Figure 4.2: A sequence of three Steiner symmetrizations transforms any quadrilateral into a rectangle.

□

We believe that the square minimizes λ_1 among rectangles of the same area.

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