



Universidad de Concepción  
Dirección de Postgrado  
Facultad de Ciencias Físicas y Matemáticas - Programa Doctorado en Matemática

# **On Some Isoperimetric Inequalities Involving Nonlocal Operators**

## **Sobre Algunas desigualdades isoperimétricas que involucran operadores no locales**

Tesis para optar al grado de Doctor en Matemática

Franco Olivares-Contador  
CONCEPCIÓN-CHILE  
2022

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# Dedicatorias

A la memoria de mi abuelo Ciro Olivares y mi abuela Oriela Azolas muertos en mis años de permanencia en el doctorado.

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# Publications

The contents of Chapter 3 appears in the work (Mahadevan and Olivares, [42]) published in 2021. The contents of Chapter 4 form part of a manuscript which is to be submitted for publication.

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# Chapter 1

## Introduction

### 1.1 Description of the problems along with some historical background

The classical isoperimetric inequality says the following: “among all regions with a given area, the circle has the smallest perimeter,” or its equivalent form: “among all regions with a given perimeter, the circle has the largest area.”

Isoperimetric inequalities restricted to classes of polygons apparently date from Zenodorus (2nd century BC). The isoperimetric inequality in the class of  $n$ -sided polygons says the following: “A regular  $n$ -gon has greater area than all other  $n$ -gons with the same perimeter” (see [6]). In one of the simplest case, we find already in Euclid (see Proposition 35, First Book of the Elements [28]) the proof that the rectangle of maximum of the area among all rectangles of given perimeter is a square.

Although these kinds of questions have been of interest since antiquity and intuitive answers to some of these questions were proposed, one can say for sure that rigorous answers to these questions were forthcoming only since the development of modern calculus. The pioneering contributions of the Bernoulli brothers, Euler, and Lagrange during the 18th century, followed by those of Weierstrass during the 19th century gave rise to the development of the area of Calculus of Variations which addresses such questions. In the 20th century it reached greater heights with the developments of more efficient tools of analysis with notable contributions by Hilbert, Noether, Tonelli, de Giorgi, and Almgren, among many others. It is often the case that, to solve a particular shape optimization problem, one has to go beyond shapes and use a mathematical framework where classical shapes form a part. The eventual answer in the relaxed setup may not be a shape but in certain problems it may turn out that a classical shape is a (or the) solution.

Lord Rayleigh, in his treatise, “The Theory of Sound” (see [48], 1894, pp. 339-340) colorfully phrased the following conjecture “If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle.” The conjecture was proved independently by Faber [22] and Krahn [35], [36] in the 1920s. The result in any dimension says that the ball minimizes the first eigenvalue of the Dirichlet Laplacian and is now referred to as the Faber-Krahn inequality (which we refer to as FKI for short). Since then many minimization or maximization problems for the principal eigenvalue of standard elliptic operators with respect to the shape of the domain (with constraints on the volume, perimeter etc.) have been studied. These results, usually, go under the

name of Faber-Krahn inequalities or in general, isoperimetric inequalities. This thesis is dedicated to the study of a few such problems. In order to put the results of the thesis in context, we mention some preceding works on such inequalities. The discussion below is by no means exhaustive.

Pólya and Szegő ([46], 1951) conjectured that among all  $N$ -gons of fixed area, the regular  $N$ -gon of the same area minimizes the first eigenvalue of the Dirichlet Laplacian. Even up until now this conjecture has only been established among the classes of triangles and quadrilaterals (see [46], [29]). For other  $N$ -gons ( $N \geq 5$ ) this still remains a challenging open problem.

Apart from the Laplacian, FKIs have been obtained for other elliptic operators, for both locally and nonlocally defined operators. Among locally defined elliptic operators, the FKI for the  $p$ -Laplacian which was first studied by Bhattacharya ([5], 1999) has been revisited in recent years in a few papers [41, 18]. The sharp lower and upper bounds for the Dirichlet Laplacian for a triangle have been studied by Siudeja [56]. Besides this, Laugesen and Siudeja have established in [37] that, among all triangles of given diameter, the equilateral triangle minimizes the sum of the first  $n$  eigenvalues of the Dirichlet Laplacian. Whereas, in the nonlocal setting, the FKI for the fractional  $p$ -Laplacian has been studied by Brasco et al. ([10], 2014) in the class of bounded open sets and by Olivares-Contador ([44], 2017) for the class of triangles and quadrilaterals for Dirichlet boundary conditions (where symmetrization techniques have been used, as in this thesis). At the same time, FKIs for the Riesz potential operator have also been studied by Rozenblum et al. ([51], 2016) and by Kalmenov et al. ([31], 2017) in the class of open bounded domains in Euclidean space. Analogous questions in other geometric settings have also been studied by Ruzhansky et al. ([54], 2016). Another functional of interest in geometric problems is the Cheeger constant of a domain which appears in Cheeger's pioneering work ([17], 1970). The Cheeger constant for a domain is the infimum of the quotient of the perimeter of a subset divided by its area, among all subsets of the domain and it also can be interpreted as the first Dirichlet eigenvalue of the 1-Laplacian (see [33], 2008). Isoperimetric estimates for the Cheeger constant were studied in Kawohl et al. ([32], 2003). In the work of D. Bucur et al. ([11], 2016) the following FKI was proved: "the regular  $N$ -gon minimizes the Cheeger constant among all  $N$ -gons with a given area". In the nonlocal setting, the fractional Cheeger problem has been studied by Brasco et al. in ([10], 2014) where, among other results, a fractional analogue of the corresponding result established in [32] was proved.

In this thesis we obtain some isoperimetric inequalities for some nonlocal shape functionals in the class of triangular and quadrilateral domains. The main tool that we use to get these isoperimetric inequalities is Steiner symmetrization.

## 1.2 Synopsis of the chapters

### Synopsis of Chapter 2

This chapter contains the technical preliminaries required for studying the problems stated above and for their proofs. In particular, we include the definition and properties of symmetric-decreasing rearrangement and Steiner symmetrization of sets and functions (Section 2.1). In that section we will especially recall Riesz's inequality for both symmetric-decreasing rearrangement and Steiner symmetrization. In Section 2.3, we give, the necessary background on the Riesz potential operator and the eigenvalue functionals



which will be studied in Chapter 3. Section 2.2 provides an introduction to the functionals which will be studied in Chapter 4. In Section 2.4, an outline of the proofs of some previously known Faber-Krahn inequalities, both local and nonlocal, will be given based on Riesz's inequality. These outlines will allow us to visualize the technique that will be used later in proving the main results. Section 2.5 gives a brief treatment of Hausdorff convergence for sets including some convergence results used later in Chapters 3 and 4.

### Synopsis of Chapter 3

Chapter 3 contains inequalities of Faber-Krahn type for the first (largest) eigenvalue of the Riesz potential operator. It is shown that the principal eigenvalue of the Riesz potential operator in the class of triangular domains of given area is maximum when the triangle is equilateral. Similarly, it is proved that the maximum value of this functional in the class of quadrilateral domains of given area is attained when the quadrilateral is a square. It is also established that the equilateral triangle and the square are, respectively, the unique maximizers.

The precise mathematical statements are to be found below. Given  $d > 1$ ,  $\Omega \subseteq \mathbb{R}^d$  a bounded domain (simply connected and open) and  $0 < \alpha < d$ , the Riesz potential operator is a compact self-adjoint operator defined on  $L^2(\Omega)$  by

$$(I_\alpha u)(x) = C(d, \alpha) \int_\Omega \frac{u(y)}{|x - y|^{d-\alpha}} dy \quad (1.2.1)$$

where  $C(d, \alpha) = \pi^{-\frac{d}{2}} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}$  is a constant given in terms of the Gamma function. The principal (largest) eigenvalue of  $I_\alpha$  admits the characterization

$$\lambda_1(\Omega) = \max \left\{ \frac{\int_\Omega \int_\Omega C(d, \alpha) \frac{u(x)u(y)}{|x - y|^{d-\alpha}} dx dy}{\int_\Omega u^2(x) dx} : u \in L^2(\Omega) \setminus \{0\} \right\}. \quad (1.2.2)$$

The main theorem of Chapter 3 is the following.

**Theorem 1.2.1.** *The maximum of  $\lambda_1(\Omega)$  among all triangles of given area is obtained when  $\Omega$  is an equilateral triangle and only when  $\Omega$  is an equilateral triangle. Similarly, the maximum of  $\lambda_1(\Omega)$  among all quadrilaterals of given area is obtained when  $\Omega$  is a square and only when  $\Omega$  is a square.*

We also briefly treat the corresponding isoperimetric inequalities for the Schatten  $p$ -norms of the Riesz potential operator for integer values of  $p$  with  $p > \max(\frac{d}{\alpha}, 2)$ . Indeed, in Section 2.3, it is shown, under this hypothesis, that the singular values of the operator form a  $p$ -summable sequence so that, by Definition 2.3.8, the operator belongs to the Schatten  $p$ -class. Note that the Riesz operator, for  $p > \frac{d}{\alpha}$ , although not in the trace class, could be Hilbert-Schmidt if  $p = 2 > \frac{d}{\alpha}$ . Moreover, being a positive compact self-adjoint operator the maximal eigenvalue is the operator norm as an operator on  $L^2$  whereas the Schatten norms are other norms of the operator containing different information on the operator.

**Theorem 1.2.2.** *Let  $p > \max(\frac{2}{\alpha}, 2)$  be a natural number. Then, the maximum of  $\|I_{\alpha, \Omega}\|_p$  among all triangles of given area is obtained when  $\Omega$  is an equilateral triangle. Similarly, the maximum of  $\|I_{\alpha, \Omega}\|_p$  among all quadrilaterals of given area is obtained when  $\Omega$  is a square.*

An introduction to the Riesz potential operator, which includes a discussion of eigenvalue functionals of interest in the thesis, is given in Section 2.3. We start Section 3.1 by giving two preliminary results which are used in the proof of the main result and are of interest in themselves. These are; firstly, the continuity of the first eigenvalue of the Riesz operator and of the Schatten norm with respect to the convergence in the Hausdorff complementary metric of a sequence of uniformly bounded convex open sets (namely, Proposition 3.1.4 and Corollary 4.2.3) and; secondly, a discussion of the equality case in Riesz's inequality for Steiner symmetrization (see Proposition 2.1.7). A proof of the latter result in dimension one appears in Lieb [40]. The equality case for this inequality with respect to the symmetric-decreasing rearrangement, in any dimension, is treated in Lieb [39]. However, a discussion of the analogous result with respect to Steiner symmetrization, for any dimension, is not easy to find.

### Synopsis of Chapter 4

In Chapter 4, the functionals under consideration are the nonlocal  $s$ -perimeter of a domain, and the corresponding Cheeger constant. It is shown that the minimum of these functionals in the class of triangular domains of given area is attained for equilateral triangles and, in the case of quadrilateral domains of given area, is attained for a square. One is able to show that equality in the isoperimetric inequality for the non-local  $s$ -perimeter in the class of triangles holds only for an equilateral triangle and likewise, is a square, in the case of quadrilaterals but similar results could not be established in the case of the nonlocal Cheeger constant.

More precisely, for  $0 < s < 1$ , for any Borel set  $E \subset \mathbb{R}^d$  the nonlocal  $s$ -perimeter  $P_s(E)$  is, by definition,

$$P_s(E) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{d+s}} dx dy. \quad (1.2.3)$$

The following scale-invariant nonlocal isoperimetric inequality

$$\frac{P_s(E)}{|E|^{1-\frac{s}{d}}} \geq \frac{P_s(B)}{|B|^{1-\frac{s}{d}}} \quad (1.2.4)$$

where  $B$  is any  $d$ -dimensional ball, holds with equality occurring if only if  $E$  is a ball (see Frank et al. [23] or Brasco et al. [10]).

For any  $\Omega \subseteq \mathbb{R}^d$ , a bounded open set, the nonlocal  $s$ -Cheeger constant is defined by

$$h_s(\Omega) = \inf_{E \subset \Omega} \frac{P_s(E)}{|E|} \quad (1.2.5)$$

over all measurable subsets  $E$  of  $\Omega$ . From (1.2.4) the following Faber-Krahn inequality for the Cheeger constant can be deduced (see Brasco et al. [10])

$$|\Omega|^{\frac{s}{d}} h_s(\Omega) \geq |B|^{\frac{s}{d}} h_s(B). \quad (1.2.6)$$

We obtain the following analogues of the isoperimetric inequalities (1.2.4) and (1.2.6) while restricting the class of domains to be triangles or quadrilaterals.

**Theorem 1.2.3.** *(a nonlocal isoperimetric inequality in the class of triangles and quadrilaterals) Let  $0 < s < 1$ . The minimum of  $P_s(\Omega)$  among all triangles (open) of given area is obtained when  $\Omega$  is an equilateral triangle and only when  $\Omega$  is an equilateral triangle. Similarly, the minimum of  $P_s(\Omega)$  among all quadrilaterals (open) of given area is obtained when  $\Omega$  is a square and only when  $\Omega$  is a square.*

**Theorem 1.2.4.** *(isoperimetric inequality for the nonlocal  $s$ -Cheeger constant for triangles and quadrilaterals) Let  $0 < s < 1$ . The minimum of  $h_s(\Omega)$  among all triangles (open) of given area is obtained when  $\Omega$  is an equilateral triangle. Similarly, the minimum of  $h_s(\Omega)$  among all quadrilaterals (open) of given area is obtained when  $\Omega$  is a square.*

Theorems 1.2.3 and 1.2.4 are proved in a similar way to Theorem 1.2.1 but this is done using, instead of the Riesz inequality for Steiner symmetrizations, the nonlocal Pólya-Szegő inequality.

# Chapter 2

## Preliminaries

### 2.1 Symmetric-decreasing rearrangement and Steiner symmetrization of sets and functions

In this subsection we recall, mainly, the notion of Steiner symmetrization and some of its properties. The definitions given below follow Lieb and Loss [40] or Brascamp, Lieb, and Luttinger [9]. We also refer to the same texts and Gruber [26] for the main properties (see [12, 3, 21, 29] for complementary information).

**Proposition 2.1.1.** (*layer-cake representation*) *Let  $\nu$  be a measure on the Borel sets of  $\mathbb{R}^+$  such that  $\phi(t) := \nu([0, t])$  is finite for every  $t > 0$ . Let  $(\Omega, \Sigma, \mu)$  be a sigma-finite measure space and  $f$  any nonnegative measurable function on  $\Omega$ . Then*

$$\int_{\Omega} \phi(f(x))\mu(dx) = \int_0^{\infty} \mu(\{x \in \Omega : f(x) > t\})\nu(dt). \quad (2.1.1)$$

In particular, if  $\phi(t) = t^p$  where  $p > 0$ , then we have

$$\|f\|_{L^p(X, d\mu)}^p = p \int_0^{\infty} t^{p-1} \mu(\{x \in \Omega : f(x) > t\})dt, \quad (2.1.2)$$

and if  $\mu$  is the Dirac measure at  $x \in \mathbb{R}^n$  and  $\phi(t) = t$ , (2.1.1) takes the form

$$f(x) = \int_0^{\infty} \chi_{\{y \in \Omega : f(y) > t\}}(x)dt = \int_0^{\infty} \chi_{\{y \in \Omega : f(y) \geq t\}}(x)dt. \quad (2.1.3)$$

**Proof:** See Theorem 1.13 of [40]. ■

The definition of the symmetric-decreasing rearrangement used here is based on the layer cake representation of a function  $f(\cdot)$  in terms of its “slices”  $\{x \in \Omega : f(x) > t\}$ .

**Definition 2.1.2. (Symmetric-decreasing rearrangement)** *For any Borel measurable subset  $\Omega \subset \mathbb{R}^d$  with finite Lebesgue measure we denote by  $\Omega^*$  the open ball centered at the origin whose measure is that of  $\Omega$ . For any nonnegative Borel measurable function  $f$  on  $\mathbb{R}^d$  vanishing at infinity (in the sense that the level sets  $\{f > t\}$  all have finite measures for any  $t > 0$ ) we define the symmetric-decreasing rearrangement  $f^*$  of  $f$  using the layer cake representation, by symmetrizing its level sets, that is*

$$f^*(x) = \int_0^{\infty} \chi_{\{y \in \mathbb{R}^d : f(y) > t\}^*}(x)dt. \quad (2.1.4)$$

$f^*$  so defined is a Borel measurable function.

**Remark 2.1.3.** *The following are the main properties of :*

1.  $f^*$  is radially symmetric and non-increasing, i.e.,  $f^*(x) = f^*(y)$  if  $|x| = |y|$  and  $f^*(x) \geq f^*(y)$  if  $|x| \leq |y|$ .
2.  $f$  and  $f^*$  are equimeasurable, i.e.,  $|\{x : f(x) > t\}| = |\{x : f^*(x) > t\}|$ .

**Steiner symmetrization:** By definition, the Steiner symmetrization of a function, in one dimension, is the same as the symmetric-decreasing rearrangement given above. The Steiner symmetrization of a function defined on a domain in  $\mathbb{R}^d$  ( $d > 1$ ) with respect to a hyperplane is given in the following definition.

**Definition 2.1.4.** *Let  $f$  be a nonnegative, Borel measurable function on  $\mathbb{R}^d$  which vanishes at infinity, and let  $H$  be any hyperplane ( $(d-1)$ -dimensional plane). We set up an orthogonal coordinate system on  $\mathbb{R}^d$  in such a way that, if  $(x', x_d) = (x_1, x_2, \dots, x_{d-1}, x_d)$  stand for the coordinates of a generic point  $x$ , then  $H$  is the plane  $x_d = 0$ .*

*A nonnegative, Borel measurable function  $\mathcal{S}f$  on  $\mathbb{R}^d$  is called the Steiner symmetrization with respect to  $H$  of  $f$ , if  $\mathcal{S}f(x_1, x_2, \dots, x_{d-1}, \cdot)$  is the symmetric-decreasing rearrangement of  $f(x_1, x_2, \dots, x_{d-1}, \cdot)$  with respect to the  $x_d$  variable, for each fixed  $x_1, \dots, x_{d-1}$ .*

It can be seen that this naturally leads to the following definition for the Steiner symmetrization of a bounded measurable set  $\Omega$  with respect to the hyperplane  $H$ .

**Definition 2.1.5. (Steiner symmetrization of a set)** *For any bounded Borel measurable set  $\Omega \subset \mathbb{R}^d$  the Steiner symmetrization of  $\Omega$  with respect to  $H$ , to be denoted by  $\mathcal{S}\Omega$ , is given by*

$$\mathcal{S}\Omega = \bigcup_{\substack{b \in H \\ \Omega \cap L_b \neq \emptyset}} \{b + t e_d : |t| \leq \frac{1}{2} |\Omega \cap L_b|\} \quad (2.1.5)$$

where  $|\Omega \cap L_b|$  is the one-dimensional Lebesgue measure of  $\Omega \cap L_b$  with  $L_b$  being the line with direction  $e_d$  passing through the point  $b$  for any  $b \in H$ .

We recall the following properties of the Steiner symmetrization of functions.

**Proposition 2.1.6.** 1. *The definitions of  $\mathcal{S}A$  and  $\mathcal{S}f$  are consistent, that is,*

$$\chi_{\mathcal{S}A} = \mathcal{S}\chi_A \text{ and } \mathcal{S}\{x : f(x) > t\} = \{\mathcal{S}f(x) > t\}.$$

*for all Borel measurable sets  $A$  with finite Lebesgue measure and for all non-negative Borel measurable functions  $f$  which vanish at infinity.*

2. *The super-level sets are equimeasurable, that is,*

$$|\{x : f(x) > t\}| = |\{\mathcal{S}f(x) > t\}|.$$

*In particular, for any measurable set  $A \subset \mathbb{R}^d$  with finite Lebesgue measure we have  $V(A) = V(\mathcal{S}A)$ .*

3. *Let  $f$  be a nonnegative Borel measurable function with  $f \in L^2(\mathbb{R}^d)$ . Then,*

$$\|f\|_2 = \|\mathcal{S}f\|_2.$$

4. Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a non-negative Borel measurable function which vanishes at infinity and let  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be continuous and monotonically increasing with  $\Phi(0) = 0$ . Then,

$$\int_{\mathbb{R}^d} \Phi(f(x))dx = \int_{\mathbb{R}^d} \Phi(\mathcal{S}f(x))dx.$$

**Proof:** For (1) see pages 183-184 of [3]. By part (1) of the Proposition 2.1.6

$$|\{x : f(x) > t\}| = |\{x : \mathcal{S}f(x) > t\}| = |\mathcal{S}\{x : f(x) > t\}|.$$

The equality  $V(A) = V(\mathcal{S}A)$  can be seen by decomposing the volume integral into one-dimensional sections perpendicular to the hyperplane  $H$  and observing that the sections have the same measure (length) before and after the rearrangement. As to (4), it is easy to see that the hypotheses on  $\Phi$  guarantee the existence of a measure  $\nu$  such that  $\Phi(t) = \nu([0, t])$  for all  $t > 0$ . Then, by Proposition 2.1.1, we get

$$\int_{\mathbb{R}^d} \Phi(f(x))dx = \int_0^\infty |\{x : f(x) > t\}|\nu(dt)$$

and

$$\int_{\mathbb{R}^d} \Phi(\mathcal{S}f(x))dx = \int_0^\infty |\{x : \mathcal{S}f(x) > t\}|\nu(dt).$$

Hence

$$\int_{\mathbb{R}^d} \Phi(f(x))dx = \int_{\mathbb{R}^d} \Phi(\mathcal{S}f(x))dx.$$

Then, (3) is just a special case of (4) for the choice  $\Phi(t) = t^2$ . ■

We now recall Riesz's inequality for the Steiner symmetrization in one dimension. This inequality for symmetric-decreasing rearrangement goes back to F. Riesz (1930, [49]). We will give the proof of this inequality from Lieb and Loss [40, Lemma 3.6] (adding some extra details) but is originally from Rogers [50] and Brascamp-Lieb-Luttinger [9]. The equality case was studied by A. Burchard [14].

**Proposition 2.1.7. (Riesz's inequality in one-dimension)** *Let  $f, g$ , and  $h$  be non-negative Borel measurable functions that vanish at infinity on  $\mathbb{R}$ , and let  $f^*, g^*$ , and  $h^*$  be their respective symmetric-decreasing rearrangements.*

*Then, for  $I(f, g, h) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(x-y)h(y) dx dy$ , we have*

$$I(f, g, h) \leq I(f^*, g^*, h^*). \quad (2.1.6)$$

**Proof:** STEP 1: Using Fubini's theorem and the layer-cake decomposition on  $f, g$ , and  $h$ , we have

$$\begin{aligned} I(f, g, h) &= \int_0^\infty \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\{f>s\}}(x)\chi_{\{g>r\}}(x-y)\chi_{\{h>t\}}(y) dx dy ds dr dt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty I(\chi_{\{f>s\}}, \chi_{\{g>r\}}, \chi_{\{h>t\}}) ds dr dt. \end{aligned} \quad (2.1.7)$$

By the same argument, we obtain

$$I(f^*, g^*, h^*) = \int_0^\infty \int_0^\infty \int_0^\infty I(\chi_{\mathcal{S}\{f>s\}}, \chi_{\mathcal{S}\{g>r\}}, \chi_{\mathcal{S}\{h>t\}}) ds dr dt. \quad (2.1.8)$$

Since  $f, g$ , and  $h$  are nonnegative functions that vanish at infinity, their respective level sets have finite measure. Thus, from (2.1.16) and (2.1.8), it can be seen that it is enough to prove (2.1.6) for characteristic functions of measurable sets of finite measure.

STEP 2: Let  $O^1, O^2$ , and  $O^3$  be sets of finite measure. By outer regularity of the Lebesgue measure, for each  $k = 1, 2, 3$ , there exists a sequence of open sets  $\{O_r^k\}_{r \in \mathbb{N}}$  (having finite measure for every  $r \in \mathbb{N}$ ) such that  $O^k \subset O_r^k \subset O_{r-1}^k$  for all  $r \in \mathbb{N}$  and  $\lim_{r \rightarrow \infty} |O_r^k| = |O^k|$ . Obviously, then we also have  $\lim_{r \rightarrow \infty} |(O_r^k)^*| = |(O^k)^*|$ . This means that  $\chi_{O_r^k} \rightarrow \chi_{O^k}$  and  $\chi_{(O_r^k)^*} \rightarrow \chi_{(O^k)^*}$  in  $L^1(\mathbb{R})$  and so, for a subsequence which we denote by the same index,  $\chi_{O_r^k} \rightarrow \chi_{O^k}$  and  $\chi_{(O_r^k)^*} \rightarrow \chi_{(O^k)^*}$  almost everywhere. The dominated convergence theorem shows that

$$\begin{aligned} \lim_{r \rightarrow \infty} I(\chi_{O_r^1}, \chi_{O_r^2}, \chi_{O_r^3}) &= I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) \quad \text{and} \\ \lim_{r \rightarrow \infty} I(\chi_{(O_r^1)^*}, \chi_{(O_r^2)^*}, \chi_{(O_r^3)^*}) &= I(\chi_{(O^1)^*}, \chi_{(O^2)^*}, \chi_{(O^3)^*}). \end{aligned}$$

So, we can further deduce that it is enough to prove (2.1.6) for characteristic functions of open sets  $O^1, O^2, O^3$  having finite measure.

STEP 3: Now, we recall that every open subset of the real line is the disjoint union of countably many open intervals. For  $k \in \{1, 2, 3\}$ , denote these intervals by  $I_1^k, I_2^k, \dots$  such that  $O^k = \cup_{n=1}^{\infty} I_n^k$ , arranged in such a way that  $|I_{n+1}^k| \leq |I_n^k|$ . If we set

$$F_m^k = \bigcup_{n=1}^m I_n^k \quad \text{with } k \in \{1, 2, 3\} \quad (2.1.9)$$

we get

$$\lim_{m \rightarrow \infty} |F_m^k| = \sum_{l=1}^{\infty} |I_l^k| = |O^k| \quad \text{for every } k \in \{1, 2, 3\}.$$

Using the monotone convergence theorem, we see that

$$\begin{aligned} \lim_{m \rightarrow \infty} I(\chi_{F_m^1}, \chi_{F_m^2}, \chi_{F_m^3}) &= I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) \quad \text{and} \\ \lim_{m \rightarrow \infty} I(\chi_{(F_m^1)^*}, \chi_{(F_m^2)^*}, \chi_{(F_m^3)^*}) &= I(\chi_{(O^1)^*}, \chi_{(O^2)^*}, \chi_{(O^3)^*}). \end{aligned}$$

The above equations further reduce the problem of showing (2.1.6) to that of establishing (2.1.6) for characteristic functions of sets  $O^1, O^2, O^3$  each of which is a finite union of disjoint open intervals of finite length.

STEP 4: Thus, we can write

$$I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) = \sum_{j=1}^J \sum_{m=1}^M \sum_{n=1}^N \int \int f_j(x - a_j) g_m(x - y - b_m) h_n(y - c_n) dx dy$$

where  $f_j, g_m$ , and  $h_n$  are characteristic functions of intervals centered at the origin and the  $a_j, b_m$ , and  $c_n$  are real numbers. We set

$$I_{jmn}(t) = \int \int f_j(x - ta_j) g_m(x - y - tb_m) h_n(y - tc_n) dx dy, \quad (2.1.10)$$

and

$$I_t(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) = \sum_{j=1}^J \sum_{m=1}^M \sum_{n=1}^N I_{jmn}(t),$$

so that  $I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) = I_1(\chi_{O^1}, \chi_{O^2}, \chi_{O^3})$ . We observe that  $I_{jmn}(t)$  is symmetric and decreasing with respect to  $t$ . Indeed, by introducing the change of variables,  $w = x - ta_j$ ,  $z = y - ta_j + tb_m$ , we obtain

$$\begin{aligned} I_{jmn}(t) &= \int \int f_j(w) g_m(w - z) h_n(z + t(a_j - b_m - c_n)) dw dz \\ &= \int u_{jm}(z) h_n(z + t(a_j - b_m - c_n)) dz. \end{aligned}$$

Then,  $u_{jm}(z) = \int f_j(w) g_m(w - z) dw$  is of the form  $|(-r, r) \cap (-s + z, s + z)|$  since  $f_j$  and  $g_m$  are characteristic functions of symmetric intervals about the origin and so,  $u_{jm}$  is a symmetric non-increasing function in  $z$ . Now,  $I_{jmn}(t) = \int u_{jm}(z) h_n(z + t(a_j - b_m - c_n)) dz$  is the integral of a symmetric non-increasing function  $u_{jm}$  on shifts of a symmetric interval and therefore, is seen to be symmetric and decreasing with respect to  $t$ .

Now, going back to (2.1.10), notice that, as  $t \rightarrow 0$ , the supporting intervals of the functions  $f_j(x - ta_j)$ ,  $g_m(x - y - tb_m)$  and  $h_n(y - tc_n)$  move toward the origin and so when the supporting intervals of any two of the  $f_j$ s or  $g_m$ s or  $h_n$ s touch each other, the couple may be replaced by the characteristic function of a unified interval. At this stage we reinitialise the sets  $O^1$ ,  $O^2$ , and  $O^3$  by using the current positions of the intervals and observe that  $I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3})$  for the reinitialised sets  $O^1$ ,  $O^2$ , and  $O^3$  is larger than it was at the beginning of Step 4. We repeat the process of Step 4 with the current values of  $O^1$ ,  $O^2$ , and  $O^3$ .

STEP 5: After a finite number of repetitions of Step 4, we obtain the desired inequality

$$I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) \leq I(\chi_{(O^1)^*}, \chi_{(O^2)^*}, \chi_{(O^3)^*}). \quad \blacksquare$$

**Corollary 2.1.8. (Riesz's inequality for Steiner symmetrization in higher dimensions)** *Let  $f$ ,  $h$ , and  $g$  be nonnegative measurable functions vanishing at infinity on  $\mathbb{R}^d$  and  $V$  a plane passing through the origin of  $\mathbb{R}^d$ . Then, taking for the definition,  $I(f, g, h) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) g(x - y) h(y) dx dy$ , we have*

$$I(f, g, h) \leq I(\mathcal{S}f, \mathcal{S}g, \mathcal{S}h). \quad (2.1.11)$$

where  $\mathcal{S}f$  is the Steiner symmetrization of  $f$  with respect to  $V$  (see Definition 2.1.4).

**Proof:** For any fixed  $z' = z_1, \dots, z_{d-1}$ ,  $f^*(z', \cdot)$ ,  $g^*(z', \cdot)$ , and  $h^*(z', \cdot)$  are, respectively, the one-dimensional symmetric-decreasing rearrangement of  $f(z', \cdot)$ ,  $g(z', \cdot)$ , and  $h(z', \cdot)$ . So, for any  $x'$ ,  $y'$  in  $\mathbb{R}^{d-1}$ , we apply Riesz's inequality for one-dimensional symmetric-decreasing rearrangement for the functions  $f(x', \cdot)$ ,  $g(x' - y', \cdot)$ , and  $h(y', \cdot)$  to obtain

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} f(x', x_d) g(x' - y', x_d - y_d) h(y', y_d) dx_d dy_d \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} f^*(x', x_d) g^*(x' - y', x_d - y_d) h^*(y', y_d) dx_d dy_d. \end{aligned}$$

Then, after integrating the inequality with respect to  $x'$ ,  $y'$ , we obtain the desired inequality.  $\blacksquare$

**Note:** An analogous result for the symmetric-decreasing rearrangement of a function in any dimension is the following Riesz's inequality

$$I(f, g, h) \leq I(f^*, g^*, h^*). \quad (2.1.12)$$



where  $f^*$  is the symmetric-decreasing rearrangement of  $f$  (see Definition 2.1.2). For a detailed treatment of this we refer to Burchard [13] and Lieb and Loss [40].

A more general rearrangement inequality published in the work of Brascamp, Lieb, and Luttinger [9] asserts the following.

**Theorem 2.1.9.** *Let  $f_1, f_2, \dots, f_m$  be non-negative functions on  $\mathbb{R}^d$  which vanish at infinity. Let  $k \leq m$  and let  $B = ((b_{ij}))$  be a  $k \times m$  matrix. If we set*

$$I(f_1, f_2, \dots, f_m) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^m f_j \left( \sum_{i=1}^k b_{ij} x_i \right) dx_1 dx_2 \dots dx_k$$

then we have

$$I(f_1, f_2, \dots, f_m) \leq I(f_1^*, f_2^*, \dots, f_m^*). \quad (2.1.13)$$

The following passage is dedicated to a discussion of the equality case of the Riesz inequality in one dimension. A proof is sketched in Lieb and Loss [40] leaving some of the work to be done by the reader. This proof is detailed in my master's thesis [45] (in Spanish). We present here a different proof given in Carlen and Maggi [15] which is based on the Brunn-Minkowski inequality. Before studying this proof, let us recall the Brunn-Minkowski inequality and the definition of a Lebesgue point of a measurable set and some relevant facts about the latter.

**Definition 2.1.10.** *We shall say that a point  $x$  is a Lebesgue point of a measurable set  $A$  in  $\mathbb{R}$ , if*

$$\lim_{\epsilon \rightarrow 0^+} \frac{|A \cap B(x, \epsilon)|}{|B(x, \epsilon)|} = 1.$$

We will denote the set of Lebesgue points of  $A$  as  $D_A$ . Clearly, by its definition, the interior of a set  $A$  is contained in  $D_A$  and  $D_A$  is contained in the closure of  $A$ . Also note that if  $x$  is a Lebesgue point of  $A$ , then  $x + y$  is a Lebesgue point of  $A + y$  and  $-x$  is a Lebesgue point of  $-A$ . Finally, it is true that  $x$  is a Lebesgue point of  $C \cap D$  if  $x$  is a Lebesgue point of  $C$  and  $x$  is a Lebesgue point of  $D$ .

**Theorem 2.1.11.** *(Lebesgue's density theorem) Let  $A \subseteq \mathbb{R}$  such that  $m(A) > 0$ . Almost every point of  $A$  is a Lebesgue point of  $A$ .*

**Proof:** See appendix D in van Rooij and Schikhof [58]. ■

The following lemma has its inspiration in Lemma 1 in A. Burchard [13].

**Lemma 2.1.12.** *Let  $A$  and  $B$  be measurable sets in  $\mathbb{R}$ ,  $\chi_A$  and  $\chi_B$  their characteristic functions. Together with the above assumptions, suppose that  $A$  and  $B$  have no isolated points. Then,*

$$\overline{A + B} = \text{supp}(\chi_A * \chi_B)$$

where  $(\chi_A * \chi_B)(x) = \int_{\mathbb{R}} \chi_A(x - y) \chi_B(y) dy$ .

**Proof:** It is clear that

$$(\chi_A * \chi_B)(x) = |(-A + x) \cap B|$$

and that, if  $\chi_A * \chi_B(x) > 0$  then  $x \in A + B$ . Therefore,

$$\text{supp}(\chi_A * \chi_B) = \overline{\{x : \chi_A * \chi_B(x) > 0\}} \subset \overline{A + B}. \quad (2.1.14)$$

Conversely, if  $x \in D_A + D_B$  then  $x = a + b$  with  $a \in D_A$  and  $b \in D_B$ . This means that  $b = x - a$  is a Lebesgue point of  $B \cap (x - A)$ . Therefore,  $|(x - A) \cap B| > 0$  which means  $x \in \text{supp}(\chi_A * \chi_B)$ . We have shown that

$$D_A + D_B \subset \text{supp}(\chi_A * \chi_B).$$

By the assumption that  $A$  and  $B$  have no isolated points and since  $D_A$  and  $A$  differ only by a set of measure zero and similarly, for  $D_B$  and  $B$ , it follows that  $D_A$  and  $A$  have the same closures and so do  $D_B$  and  $B$ . Therefore, since the support of a function is a closed set, this implies that

$$\overline{D_A + D_B} = \overline{A + B} \subset \text{supp}(\chi_A * \chi_B). \quad (2.1.15)$$

So, we obtain the desired conclusion from (2.1.14) and (2.1.15).  $\blacksquare$

The Brunn-Minkowski inequality in one dimension (see [27]) states that

$$|A + B| \geq |A| + |B|,$$

with  $A$  and  $B$  two nonempty measurable sets in  $\mathbb{R}$ . There is equality if and only if either  $A$  consists of a single point or  $B$  consists of a single point, or  $A$  and  $B$  are of the form  $A = \bar{I} \setminus N_A$  and  $B = \bar{J} \setminus N_B$  where  $N_A$  and  $N_B$  are sets of measure zero, and  $\bar{I}$  and  $\bar{J}$  are closed intervals.

**Proposition 2.1.13.** (*Lieb's theorem on cases of equality in the Riesz rearrangement inequality*). *Let  $f$ ,  $g$ , and  $h$  be non-negative and non-zero integrable functions that vanish at infinity on  $\mathbb{R}$  where  $g$  is symmetric and strictly decreasing. Then there is equality in (2.1.6) if and only if  $f(x) = f^*(x - a)$  and  $h(x) = h^*(x - a)$  for some  $a \in \mathbb{R}$ .*

**Proof:** Let  $F_r = \{f > r\}$ ,  $G_s = \{g \geq s\}$  and  $H_t = \{h > t\}$ . Since we may write

$$I(f, g, h) = \int_0^\infty \int_0^\infty \int_0^\infty I(\chi_{F_r}, \chi_{G_s}, \chi_{H_t}) ds dr dt$$

and

$$I(f^*, g^*, h^*) = \int_0^\infty \int_0^\infty \int_0^\infty I(\chi_{F_r^*}, \chi_{G_s^*}, \chi_{H_t^*}) ds dr dt,$$

under the hypothesis that  $g = g^*$ , equality in Riesz's inequality implies that

$$I(\chi_{F_r}, \chi_{G_s}, \chi_{H_t}) = I(\chi_{F_r^*}, \chi_{G_s}, \chi_{H_t^*}) \text{ a.e. } r, s, t. \quad (2.1.16)$$

Our aim is to show that, for almost all  $r$ ,  $F_r$  is an interval and they are all symmetric with respect to some  $a$  and similarly, for almost all  $t$ ,  $H_t$  is an interval and they are all symmetric intervals about the same  $a$ .

Since  $g$  is symmetric and strictly decreasing about intervals for every  $l$  with  $0 \leq l$  there exists  $s$  such that  $[-l, l] = \{g \geq s\}$ . For example  $s$  may taken to be  $s = \sup\{h : \{g \geq h\} \subset [-l, l]\}$ . So, for any given  $r$  and  $t$ , assuming without loss of generality that  $|F_r| \leq |H_t|$  we can find  $s$  such that

$$(F_r)^* + G_s = (H_t)^*. \quad (2.1.17)$$

That is, by Lemma 2.1.12,  $(H_t)^*$  and  $\text{supp}(\chi_{F_r^*} \star \chi_{G_s}(x))$  coincide except for a null set and so,

$$\begin{aligned} I(\chi_{F_r^*}, \chi_{G_s}, \chi_{H_t^*}) &= \int_{\mathbb{R}} (\chi_{F_r^*} \star \chi_{G_s})(x) \chi_{H_t^*}(x) dx \\ &= \int_{\mathbb{R}} (\chi_{F_r^*} \star \chi_{G_s})(x) dx \\ &= |(F_r)^*| |G_s| = |F_r| |G_s| \end{aligned} \quad (2.1.18)$$

If it happens that  $F_r$  is not an interval and since  $G_s$  is an interval, then by the equality condition for Brunn Minkowski inequality, necesarrily one must have

$$|F_r + G_s| > |F_r| + |G_s| = |(F_r)^*| + |G_s| = |(F_r)^* + G_s| = |(H_t)^*| = |H_t|. \quad (2.1.19)$$

Therefore,  $\{x \notin H_t, x \in \text{supp}(\chi_{F_r} \star \chi_{G_s})(x)\}$  is of positive measure and so,

$$\begin{aligned} 0 < \int_{\mathbb{R}} (\chi_{F_r} \star \chi_{G_s})(x) \chi_{(H_t)^c}(x) dx &= \int_{\mathbb{R}} \chi_{F_r} \star \chi_{G_s}(x) dx - \int_{\mathbb{R}} (\chi_{F_r^*} \star \chi_{G_s})(x) \chi_{H_t}(x) dx \\ &= |F_r| |G_s| - \int_{\mathbb{R}} (\chi_{F_r} \star \chi_{G_s})(x) \chi_{H_t}(x) dx. \end{aligned} \quad (2.1.20)$$

So, by using (2.1.18) in (2.1.20) we get

$$I(\chi_{F_r}, \chi_{G_s}, \chi_{H_t}) < I(\chi_{F_r^*}, \chi_{G_s}, \chi_{H_t^*})$$

and by the continuity of the integrals on either side with respect to  $r$ ,  $s$ , and  $t$ , the inequality has to hold on an open set of values for  $r$ ,  $s$ , and  $t$  contradicting (2.1.16). So,  $F_r$  and  $H_t$  are intervals (up to sets of measure zero). We shall now prove that,  $F_r := A$  and  $H_t := B$  are centered at the same point. For this purpose, let us write (2.1.16) as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(x) \chi_{(-l, l)}(x - y) \chi_B(y) dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{A^*}(x) \chi_{(-l, l)}(x - y) \chi_{B^*}(y) dx dy, \quad (2.1.21)$$

for  $l > 0$ . Let's consider that  $A$  has center  $a$  and  $B$  has center  $b$ . Suppose that  $a \neq b$ . Without loss of generality, we assume that  $b > a$ . Using the above, the equation (2.1.21) is equivalent to

$$\int_{B^*} |A^* \cap [(-l, l) + y]| dy = \int_{B^*} |A^* \cap [(-l, l) + y + b - a]| dy. \quad (2.1.22)$$

From step 4 of the proof of the Proposition 2.1.7 we know that  $|A^* \cap [(-l, l) + y]|$  is a non-increasing function of  $y$  (this function is also a continuous function), in particular this function is strictly decreasing in  $y \in \left(\frac{|A|}{2} - l, \frac{|A|}{2} + l\right)$  and hence injective. Using the above and (2.1.22) we have

$$|A^* \cap [(-l, l) + y]| = |A^* \cap [(-l, l) + y + b - a]|. \quad (2.1.23)$$

Therefore, by the injectivity in  $\left(\frac{|A|}{2} - l, \frac{|A|}{2} + l\right)$  we get  $a = b$ .

Fixing  $r_0$  and varying  $t$  (and vice versa) shows that the center will not depend on  $r$  or  $t$ . So almost all intervals  $\{h > t\}$  have the same center (the center of  $\{f > r_0\}$ ). This means  $h = h^*$  except for a translation.

In the same way, fixing  $t$  and varying  $r$  we conclude as before that  $f = f^*$  except for a translation. ■

We end this subsection by mentioning the following properties of the Steiner symmetrization of sets. Let us first fix some notions. A convex body is a compact convex set with nonempty interior. By  $\mathcal{K}^d$  we denote the set of all convex bodies in  $\mathbb{R}^d$ . For a convex body  $A$  in  $\mathbb{R}^d$ , the inradius  $r(A)$  is the supremum of the radii of open balls contained in  $A$  and the circumradius  $R(A)$  is the infimum of the radii of open balls containing  $A$ .

**Proposition 2.1.14.** *Let  $A, B \in \mathcal{K}^d$ . Then,*

1.  $\mathcal{S}A \subseteq \mathcal{S}B$  for  $A \subseteq B$ .
2.  $r(A) \leq r(\mathcal{S}A)$ .
3.  $R(\mathcal{S}A) \leq R(A)$ .
4.  $S(\mathcal{S}A) \leq S(A)$  where  $S(A)$  denotes the surface area (perimeter) of  $A$ .

**Proof:** We refer to Gruber [26, Proposition 9.1] for a proof. ■

## 2.2 The fractional perimeter, the fractional Cheeger constant, and fractional order spaces

Our aim here is to provide the definitions and some main properties of the nonlocal perimeter and nonlocal Cheeger constant of sets.

**Perimeter functionals:** We first recall that a Borel subset  $\Omega$  of  $\mathbb{R}^d$  is said to be of finite perimeter if

$$P(\Omega) := \sup \left\{ \int_{\Omega} \operatorname{div} v \, dx : v \in C_c^1(\Omega; \mathbb{R}^d), \|v\|_{\infty} \leq 1 \right\} < \infty.$$

For  $0 < s < 1$ , for any Borel set  $E \subset \mathbb{R}^d$  the nonlocal  $s$ -perimeter  $P_s(E)$  is, by definition,

$$P_s(E) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{d+s}} dx dy. \quad (2.2.1)$$

The right hand side is just the Gagliardo semi-norm  $[\chi_E]_{W^{s,1}}$  of  $\chi_E$ . In general, for  $1 \leq p < \infty$  the fractional order space  $\widetilde{W}_0^{s,p}(\Omega)$  is taken to be the closure of  $C_0^\infty(\Omega)$  in the fractional order Sobolev space  $W^{s,p}(\mathbb{R}^d)$  equipped with the norm

$$\left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} dx dy + \int_{\mathbb{R}^d} |u|^p(x) \, dx \right)^{1/p}.$$

**Example 2.2.1.** For  $0 < s < 1$ , a direct computation gives the nonlocal perimeter of an interval  $(-a, a)$  with  $a > 0$

$$\begin{aligned} P_s((-a, a)) &= 2 \int_{-a}^a \left( \int_{-\infty}^{-a} \frac{dx}{|x-y|^{1+s}} + \int_a^{+\infty} \frac{dx}{|x-y|^{1+s}} \right) dy \\ &= 4 \int_{-a}^a \int_a^{+\infty} \frac{1}{|x-y|^{1+s}} dx dy \\ &= \frac{4(2a)^{1-s}}{s(1-s)}. \end{aligned}$$

**Example 2.2.2.** For  $0 < s$ , using spherical coordinates one can explicitly calculate the nonlocal perimeter of a ball  $B(0, R)$ ,

$$P_s(B(0, R)) = P_s(B(x, R)) = 2 \int_{B(x, R)} \int_{B(x, R)^c} \frac{1}{|x-y|^{d+s}} dx dy = \frac{2\omega_{d-1}\omega_d R^{d-s}}{s}.$$

We refer to Mazya et al. [43] and Bourgain et al. [8], respectively, for the following asymptotic behavior as  $s \rightarrow 0^+$  and  $s \rightarrow 1^-$  of the fractional perimeter:

$$\lim_{s \rightarrow 0^+} s P_s(\Omega) = C(d)|\Omega| \quad \text{and} \quad \lim_{s \rightarrow 1^-} (1-s) P_s(\Omega) = C'(d)P(\Omega), \quad (2.2.2)$$

where  $C(d)$  and  $C'(d)$  are constants.

So, the fractional perimeter is intermediate between the perimeter and the Lebesgue measure. An interpolation result between the  $s$ -perimeter  $P_s(\cdot)$  and the classical perimeter  $P(\cdot)$  is given below.

**Proposition 2.2.3.** Let  $s \in (0, 1)$ , for every finite perimeter set  $E \subset \mathbb{R}^d$  we have

$$P_s(E) \leq \frac{2^{1-s}}{(1-s)s} P(E)^s |E|^{1-s} \quad (2.2.3)$$

**Proof:** This is a special case of Proposition 4.2 [10] given as Corollary 4.4 therein. ■

**Note:** The above Proposition 2.2.3 says, in particular that, if a set  $E$  has finite perimeter  $P(E)$  then its fractional perimeter  $P_s(E)$  is also finite.

**Cheeger constants:** The Cheeger constant of a set  $\Omega \subseteq \mathbb{R}^d$  has the following definition

$$h_1(\Omega) = \inf_{A \subseteq \Omega} \frac{P(A)}{|A|}. \quad (2.2.4)$$

The Cheeger constant, which appears in a well-known work of Cheeger [17], is an important quantity and appears in many different contexts. For a recent overview of this theme we refer to Leonardi [38]. Here, we are interested in some questions involving a nonlocal version of the Cheeger constant.

For any  $\Omega \subseteq \mathbb{R}^d$ , a bounded open set, the nonlocal  $s$ -Cheeger constant is defined by

$$h_s(\Omega) = \inf_{E \subset \Omega} \frac{P_s(E)}{|E|}, \quad (2.2.5)$$

over all measurable subsets  $E$  of  $\Omega$ . The existence of a measurable set  $E_\Omega \subset \Omega$  achieving the infimum in (2.2.5) has been shown in Brasco et al. [10, Proposition 5.3]. A minimizer for  $h_s(\Omega)$  is said to be an  $s$ -Cheeger set of  $\Omega$ .

The following properties are easy to prove starting from the definition.

**Proposition 2.2.4.** *Let  $\Omega \subset \mathbb{R}^d$  be an open and bounded set.*

1. (homothety law)  $P_s(k\Omega) = k^{d-s}P_s(\Omega)$  and  $h_s(k\Omega) = k^{-s}h_s(\Omega)$  for  $k > 0$ .
2. (translation invariance) We have  $P_s(\Omega + x) = P_s(\Omega)$  and  $h_s(\Omega + x) = h_s(\Omega)$  for any  $x \in \mathbb{R}^d$ .
3. (rotation invariance) We have  $P_s(R\Omega) = P_s(\Omega)$  and  $h_s(R\Omega) = h_s(\Omega)$  for any isometry  $R$ .
4. (domain monotonicity)  $h_s(A) \geq h_s(B)$  for bounded open sets  $A, B$  such that  $A \subseteq B$ .

**Note:** Note that the perimeter functional has, in general, no monotonicity properties with respect to domain inclusions.

**Definition 2.2.5.** *It is said that  $\Omega$  is  $s$ -calibrable if it is an  $s$ -Cheeger set of itself, that is,*

$$h_s(\Omega) = \frac{P_s(\Omega)}{|\Omega|}.$$

**Example 2.2.6.** *Every ball  $B := B(0, R)$  of  $\mathbb{R}^d$  is  $s$ -calibrable. This is a direct consequence of the isoperimetric inequality (1.2.4) which is proved in Brasco et al. [10], which gives for every  $E \subset B$*

$$\frac{P_s(E)}{|E|} \geq \frac{P_s(B)}{|B|} \left( \frac{|B|}{|E|} \right)^{\frac{s}{d}} \geq \frac{P_s(B)}{|B|},$$

and so

$$h_s(B) = \frac{P_s(B)}{|B|}.$$

Using the Example 2.2.2 we have  $h_s(B) = \frac{2\omega_{d-1}R^{-s}}{s}$ .

As compared to the local Cheeger constant, several issues around the nonlocality of  $s$ -Cheeger constants are not yet well understood. For example, while in the local setting for any convex set the Cheeger set is known to be unique [16], it is not known if this is true in the nonlocal setting. These are not, however, the questions that we will be interested in, in this thesis, but, rather, we are interested in some isoperimetric inequalities involving these quantities.

## 2.3 Riesz operators

In this subsection we give a brief introduction to the Riesz potential operator and discuss some spectral functionals which are of interest to us. The main references for this section are Stein [57], Vladimirov [59], and articles by Ruzhansky et al. [51, 53, 54].

Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded domain (simply connected, open) and  $d > 1$ . Let  $0 < \alpha < d$ . The Riesz potential operator on  $L^2(\Omega)$  is defined by

$$(I_{\alpha, \Omega} u)(x) = C(d, \alpha) \int_{\Omega} \frac{u(y)}{|x - y|^{d-\alpha}} dy \quad (2.3.1)$$

with  $C(d, \alpha) = \pi^{-\frac{d}{2}} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}$ . If  $2\alpha < d$ , for  $u \in L^2(\Omega)$ , the Riesz potential  $I_{\alpha}(u)$  is well defined almost everywhere.

**Note:** On  $\mathbb{R}^d$ , the Riesz potential operator  $I_{\alpha, \mathbb{R}^d}$  is the inverse of the fractional Laplacian operator  $(-\Delta)^s$ , with  $\alpha = 2s$ , which may be defined for smooth functions as follows:

$$(-\Delta)^s u(x) := a(d, s) \lim_{\delta \rightarrow 0^+} \int_{\{y \in \mathbb{R}^n : \delta \leq |x-y|\}} \frac{(u(x) - u(y))}{|x-y|^{n+2s}} dy$$

where  $a(d, s)$  is a suitably defined constant.

**Proposition 2.3.1.** *Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded domain,  $d > 1$ . For  $0 < \alpha < d$ , the Riesz potential operator  $I_{\alpha, \Omega}$  is a bounded linear map from  $L^2(\Omega)$  to  $L^2(\Omega)$ .*

**Proof:** The proof given here is extracted from Section 15.4 of Vladimirov [59]. Let  $R$  be large enough such that  $\Omega \subseteq \overline{B}(0, R)$ . Then, using the Cauchy-Schwarz inequality followed by Tonelli's theorem, we obtain the following estimate

$$\begin{aligned} \|I_{\alpha, \Omega} u\|_{L^2(\Omega)}^2 &= \int_{\Omega} |I_{\alpha, \Omega} u(x)|^2 dx \\ &= \int_{\Omega} \left| \int_{\Omega} \frac{C(d, \alpha) u(y)}{|x-y|^{d-\alpha}} dy \right|^2 dx \\ &= (C(d, \alpha))^2 \int_{\Omega} \left| \int_{\Omega} \left( \frac{1}{|x-y|^{(d-\alpha)/2}} \right) \left( \frac{u(y)}{|x-y|^{(d-\alpha)/2}} \right) dy \right|^2 dx \\ &\leq (C(d, \alpha))^2 \int_{\Omega} \left( \int_{\Omega} \frac{1}{|x-y'|^{d-\alpha}} dy' \right) \left( \int_{\Omega} \frac{|u(y)|^2}{|x-y|^{d-\alpha}} dy \right) dx \\ &\leq (C(d, \alpha))^2 \max_{x \in \overline{B}(0, R)} \left( \int_{\overline{B}(0, R)} \frac{1}{|x-y'|^{d-\alpha}} dy' \right) \max_{y \in \overline{B}(0, R)} \left( \int_{\overline{B}(0, R)} \frac{1}{|x-y|^{d-\alpha}} dx \right) \int_{\Omega} |u(y)|^2 dy \\ &\leq (C(d, \alpha))^2 N^2 \int_{\Omega} |u(y)|^2 dy < \infty \end{aligned}$$

where it is enough to take  $N = \int_{\overline{B}(0, 2R)} \frac{1}{|x|^{d-\alpha}} dx$  (being finite if  $\alpha > 0$ ). This shows that  $I_{\alpha, \Omega}$  is a bounded operator on  $L^2(\Omega)$ . ■

**Proposition 2.3.2.** For  $0 < \alpha, \beta$  and  $\alpha + \beta < d$ , the Riesz potential operator satisfies the semigroup property

$$I_\alpha I_\beta = I_{\alpha+\beta}, \quad (2.3.2)$$

with  $I_\alpha := I_{\alpha, \mathbb{R}^d}$ .

**Proof:** See Stein [57, Chapter 5, Section 1]. ■

**Proposition 2.3.3.** Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded domain where  $d > 1$ . For  $0 < \alpha < d$ , the Riesz potential operator  $I_{\alpha, \Omega}$  is a nonnegative operator.

**Proof:** Since the kernel of the operator  $I_\alpha$  is symmetric, being a bounded symmetric operator it is self-adjoint.

The non-negativity can be deduced using the semigroup property (2.3.2) argument from Proposition 2.1 of [51]. Consider  $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  defined by

$$T(u)(x) = C(d, \alpha/2) \int_{\mathbb{R}^d} \frac{\chi_\Omega(y)u(y)}{|x-y|^{d-\frac{\alpha}{2}}} dy, \quad (2.3.3)$$

and its adjoint is given by

$$T^*(u)(y) = \chi_\Omega(y) C(d, \alpha/2) \int_{\mathbb{R}^d} \frac{u(x)}{|x-y|^{d-\frac{\alpha}{2}}} dx. \quad (2.3.4)$$

Then, we note that  $T^*T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  is given by

$$C(d, \alpha) \chi_\Omega(x) \int_{\mathbb{R}^d} \frac{\chi_\Omega(y)u(y)}{|x-y|^{d-\alpha}} dy,$$

by using the semi-group property (2.3.2). We shall denote  $T^*T$  by  $\tilde{I}_{\alpha, d}$  which is, clearly, a non-negative operator. Finally, we note that under the direct sum decomposition  $L^2(\mathbb{R}^d) = L^2(\Omega) \oplus L^2(\mathbb{R}^d \setminus \Omega)$ , it's clear that  $\tilde{I}_{\alpha, d} \cong I_{\alpha, \Omega} \oplus \mathbf{0}$ . Therefore, the component  $I_{\alpha, \Omega}$  is a non-negative operator. ■

We recall a general result on the compactness of an integral operator on  $L^2$  space which will give us the compactness of the Riesz operator. We refer the reader to Cwikel [19] for details of this result while we limit ourselves to the description of the result in what follows. For this we first recall the definition of the weak  $L^p$  space.

**Definition 2.3.4.** Let  $(X, \mu)$  be a  $\sigma$ -finite measure space. For  $0 < p < \infty$ , the space of weak  $L^p$  functions on  $X$ , denoted by  $L^{p, \infty}(X, \mu)$  (or weak- $L^p(X, \mu)$ ) is the set of equivalence classes of  $\mu$ -almost everywhere equal functions for which

$$\|f\|_{L^{p, \infty}(X)} := \sup(\{\alpha^p \mu_f(\alpha) : \alpha > 0\})^{\frac{1}{p}}$$

is finite, where  $\mu_f(\alpha) = \mu(\{|f| \geq \alpha\})$  is the  $\mu$  measure of the super-level set  $\{|f| \geq \alpha\}$ .

**Note:** For any  $0 < p < \infty$  we have the continuous inclusion  $L^p(X, \mu) \subset L^{p, \infty}(X, \mu)$  which follows from Chebyshev's inequality

$$\alpha^p \mu_u(\alpha) = \int_{\mathbb{R}^d} \alpha^p \chi_{\{|u| > \alpha\}} d\mu \leq \int_{\mathbb{R}^d} |u(x)|^p \chi_{\{|u| > \alpha\}} d\mu \leq \int_{\mathbb{R}^d} |u(x)|^p d\mu.$$



**Example 2.3.5.** For  $0 < p < \infty$ , it is easy to check that  $u(x) := \frac{1}{|x|^{\frac{d}{p}}}$  belong to  $L^{p,\infty}(\mathbb{R}^d)$  but not does not belong to  $L^p(\mathbb{R}^d)$ . This shows that the inclusion  $L^p(X, \mu) \subset L^{p,\infty}(X, \mu)$  is, in general, strict.

**Proposition 2.3.6.** Let  $2 < p < \infty$  and let  $p'$  be the conjugate exponent satisfying  $\frac{1}{p} + \frac{1}{p'} = 1$ . Given  $g \in L^p(\mathbb{R}^d)$  and  $f \in \text{weak-}L^{p'}(\mathbb{R}^d)$  let us define, for any  $u \in L^2(\mathbb{R}^d)$ ,

$$B_{f,g}u(x) := \int_{\mathbb{R}^d} f(x-y)g(y)u(y)dy.$$

Then,  $B_{f,g}$  is a compact operator on  $L^2(\mathbb{R}^d)$ . In particular its singular values  $s_k$  (the eigenvalues of  $((B_{f,g})^*B_{f,g})^{1/2}$ ) satisfy:

$$\sup_{k \geq 1} k^{\frac{1}{p}} s_k(B_{f,g}) \leq C_p \|f\|_{L^{p',\infty}} \|g\|_{L^p}. \quad (2.3.5)$$

**Proof:** See the estimate (1) of Cwikel [19]. ■

**Proposition 2.3.7.** Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded domain,  $d > 1$ . For  $0 < \alpha < d$ , the Riesz potential operator  $I_{\alpha,\Omega}$  is a compact operator. The eigenvalues of  $I_{\alpha,\Omega}$  satisfy the estimates

$$\lambda_j(\Omega) \leq C j^{-\alpha/d} |\Omega|^{\frac{\alpha}{2d}}. \quad (2.3.6)$$

**Proof:** We follow [51, Proposition 2.1] here. We apply Proposition 2.3.6, while choosing  $f = \frac{1}{|\cdot|^{\frac{d-\alpha}{2}}}$  which is in weak  $L^{\frac{2d}{2d-\alpha}}(\mathbb{R}^d)$  and  $g = \chi_{\Omega}$  which is in  $L^{\frac{2d}{\alpha}}(\mathbb{R}^d)$ , to deduce that  $T$  defined by (2.3.3) is a compact operator. Therefore the operator  $T^*T$  is compact, but this is the operator  $\tilde{I}_{\alpha,d}$  as we have seen during the proof of Proposition 2.3.7 and consequently, the summand  $I_{\alpha,\Omega}$  is a compact operator.

The estimate (2.3.5) then yields

$$j^{\frac{\alpha}{2d}} s_j(T) \leq C |\Omega|^{\frac{\alpha}{2d}}. \quad (2.3.7)$$

Since the eigenvalues of  $I_{\alpha,\Omega}$  equal the squares of the singular numbers of  $T$ , we get

$$\lambda_j(I_{\alpha,\Omega}) = s_j(T)^2 \leq K j^{-\frac{\alpha}{d}} |\Omega|^{\frac{\alpha}{d}}. \quad (2.3.8)$$

We recall the following definition of operators of Schatten class from [53] and refer the reader to the details therein for more information. ■

**Definition 2.3.8.** Let  $1 \leq p < \infty$ . A compact operator  $T : H \rightarrow H$  on a Hilbert space  $H$  belongs to the Schatten class  $S_p(H)$  if

$$\|T\|_p = \left( \sum_{k=1}^{\infty} s_k^p \right)^{\frac{1}{p}} < \infty$$

where  $s_k$  are the singular number of  $T$ .  $\|T\|_p$  is called the Schatten-von Neumann norm of  $T$ . When  $T$  is self-adjoint and belongs to  $S_p(H)$  for  $p = 2$ , it is called a Hilbert-Schmidt operator and when true for  $p = 1$  it is said to be of trace-class. If  $T$  is a self-adjoint operator in  $S_p(H)$  and  $p$  is a positive integer, notice that  $\|T\|_p^p$  is just  $\text{Tr } T^p$ .

**Corollary 2.3.9.** *The operator  $I_{\alpha,\Omega}$  belongs to the Schatten class  $S_p(L^2(\Omega))$  for  $p > \frac{d}{\alpha}$ .*

**Proof:** This follows immediately from the estimate (2.3.6) and the definition of  $S_p(L^2(\Omega))$ . ■

It will also be useful to keep in mind the following result (see Goffeng [25, Theorem 2.4]) which allows to give an alternate description of the Schatten  $p$ -norm for integer values of  $p (> 2)$ . Given,  $1 \leq p, q < \infty$ , take  $L^{p,q}$  to be defined as consisting of the class of functions  $k(x, y)$  on  $\Omega \times \Omega$  such that

$$\|k\|_{L^{p,q}} = \left( \int_{\Omega} \left( \int_{\Omega} |k(x, y)|^p dx \right)^{\frac{q}{p}} dy \right)^{\frac{1}{q}} < \infty.$$

The Hermitian conjugate of the function  $k$  is defined by  $k^*(x, y) := \overline{k(y, x)}$ .

**Theorem 2.3.10.** *Suppose that  $K_j : L^2(\Omega) \rightarrow L^2(\Omega)$  are operators with integral kernels  $k_j$  for  $j = 1, \dots, m$  such that  $\|k_j\|_{L^{p',p}}, \|k_j^*\|_{L^{p',p}} < \infty$  for certain  $p > 2$  where  $p'$  denotes the exponent conjugate of  $p$ . Then, for  $m \geq p$ , the operator  $K_1 K_2 \dots K_m$  is a trace class operator, and we have the trace formula*

$$Tr(K_1 K_2 \dots K_m) = \int_{\Omega^m} \prod_{j=1}^m k_j(x_j, x_{j+1}) dx_1 dx_2 \dots dx_m$$

where we identify  $x_{m+1}$  with  $x_1$ .

As an immediate corollary we have.

**Corollary 2.3.11.** *Let  $p > \max\left(\frac{d}{\alpha}, 2\right)$  be a natural number. Then, we have*

$$\|I_{\alpha,\Omega}\|_p^p = \sum_{k=1}^{\infty} s_k^p = Tr(I_{\alpha,\Omega}^p) = (C(d, \alpha))^p \int_{\Omega^p} \prod_{k=1}^p |x_k - x_{k+1}|^{\alpha-d} dx_1 \dots dx_p, \quad (2.3.9)$$

where  $x_{p+1}$  is identified with  $x_1$ .

**Proof:** The hypothesis on  $p$  guarantees that  $|\cdot|^{-(d-\alpha)}$  is locally in  $L^{p'}$  allowing us to apply Theorem 2.3.10 with  $m = p$  and obtain the desired conclusion. ■

The spectral functionals of interest in the thesis are, the first eigenvalue  $\lambda_1(\Omega)$  of the Riesz operator and the Schatten norm  $\|I_{\alpha,\Omega}\|_p$  of the Riesz operator (for  $p$  in  $\mathbb{N}$ ). We gather below some of their relevant properties which will be used in Chapters 3 and 4. The first (largest) eigenvalue of the Riesz operator is characterized as follows:

$$\lambda_1(\Omega) = \max \left\{ \frac{\int_{\Omega} \int_{\Omega} C(d, \alpha) \frac{u(x)u(y)}{|x-y|^{d-\alpha}} dx dy}{\int_{\Omega} u^2(x) dx} : u \in L^2(\Omega) \setminus \{0\} \right\}. \quad (2.3.10)$$

It can be shown that the maximizer  $u_1$  for (1.2.2) exists, and is, in fact, continuous and satisfies the following Euler-Lagrange equation in the weak form

$$\int_{\Omega} \int_{\Omega} C(d, \alpha) \frac{u_1(x)\phi(y)}{|x-y|^{d-\alpha}} dx dy = \lambda_1(\Omega) \int_{\Omega} u_1(x)\phi(x) dx \quad \text{for all } \phi \in L^2(\Omega), \quad (2.3.11)$$

the strong formulation being

$$C(d, \alpha) \int_{\Omega} \frac{u_1(y)}{|x-y|^{d-\alpha}} dy = \lambda_1(\Omega) u_1(x) \quad \text{in } L^2(\Omega). \quad (2.3.12)$$

Moreover, the first eigenvalue is simple and the eigenfunction is of constant sign as stated in Lemma 3.1 of [54] and referred to as Jentzsch's theorem.

**Proposition 2.3.12.** *Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded domain,  $d > 1$ . For  $0 < \alpha < d$ , the first eigenvalue  $\lambda_1$  (the largest) of the Riesz potential operator  $I_{\alpha, \Omega}$  is positive and simple; the corresponding eigenfunction  $u_1$  can be chosen positive.*

**Proof:** We reproduce the following proof from Lemma 3.1 of [54]. It will use the fact that  $I_{\alpha, \Omega}^2$ , being the square of  $I_{\alpha, \Omega}$ , is also a compact, self-adjoint, and positive linear operator, and by the variational principle we have that

$$\lambda_1^2 = \sup_{f \in L^2(\Omega), f \neq 0} \frac{\langle I_{\alpha, \Omega}^2 f, f \rangle}{\|f\|^2}. \quad (2.3.13)$$

Let  $\lambda_1$  be the first eigenvalue of  $I_{\alpha, \Omega}$  and  $u_1$  an eigenfunction corresponding to  $\lambda_1$ , that is,  $I_{\alpha, \Omega} u_1 = \lambda_1 u_1$ . We keep in mind that  $I_{\alpha, \Omega}^2 u_1 = \lambda_1^2 u_1$ . We assume, without loss of generality, that the  $L^2$  norm of  $u_1$  is 1.

First, we shall prove that  $u_1$  cannot change sign in the domain  $\Omega$ , which is equivalent to,

$$|u_1(x)u_1(y)| = u_1(x)u_1(y) \quad \text{for all } x \in \Omega \text{ and } y \in \Omega. \quad (2.3.14)$$

We can proceed by assuming that (2.3.14) is false, so there are  $x_0$  and  $y_0 \in \Omega$  such that

$$|u_1(x_0)u_1(y_0)| > u_1(x_0)u_1(y_0). \quad (2.3.15)$$

From the continuity of  $u_1$ , there are open neighborhoods  $U(x_0, r) \subset \Omega$  and  $U(y_0, r) \subset \Omega$  such that

$$|u_1(x)u_1(y)| > u_1(x)u_1(y) \quad \text{for all } x \in U(x_0, r) \text{ and } y \in U(y_0, r).$$

Therefore,

$$\begin{aligned} \langle I_{\alpha, \Omega}^2 |u_1|, |u_1| \rangle &= \langle I_{\alpha, \Omega} |u_1|, I_{\alpha, \Omega} |u_1| \rangle = C(d, \alpha)^2 \int_{\Omega} \int_{\Omega} \int_{\Omega} \frac{|u_1|(y)}{|y-z|^{d-\alpha}} \frac{|u_1|(x)}{|x-z|^{d-\alpha}} dy dx dz \\ &> C(d, \alpha)^2 \int_{\Omega} \int_{\Omega} \int_{\Omega} \frac{u_1(y)}{|y-z|^{d-\alpha}} \frac{u_1(x)}{|x-z|^{d-\alpha}} dy dx dz \\ &= \langle I_{\alpha, \Omega} u_1, I_{\alpha, \Omega} u_1 \rangle = \lambda_1^2. \end{aligned} \quad (2.3.16)$$

which contradicts the variational principle (2.3.13). So,  $u_1$  is of the same sign in  $\Omega$  and moreover, there cannot be any  $x_0$  in  $\Omega$  such that  $u(x_0) = 0$ . Indeed, it follows from the relation

$$u_1(x_0) = \frac{1}{\lambda_1} \pi^{-\frac{d}{2}} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} \int_{\Omega} \frac{u(y)}{|x_0-y|^{d-\alpha}} dy$$

that if  $u_1(x_0) = 0$  then it is zero identically in the whole of  $\Omega$  which is not possible.

Finally, we show that  $\lambda_1$  is simple. Let's suppose that  $u_1$  and  $f_1$  are two eigenfunctions corresponding to  $\lambda_1$  which are linearly independent, then  $u_1 + cf_1$  is also an eigenfunction for  $\lambda_1$  for an arbitrary real number  $c$ . By what has been proved, without loss of generality  $u_1 + cf_1 > 0$  in  $\Omega$ , that is,  $c > -\frac{u_1}{f_1}$ . This is a contradiction since  $c$  is an arbitrary real number. ■

**Proposition 2.3.13.** *Let  $\Omega \subset \mathbb{R}^d$  be an open and bounded set. Then, the domain functionals  $\lambda_1(\cdot)$  (the first eigenvalue of the Riesz operator) and  $\|I_{\alpha,\cdot}\|_p$  (the Schatten norm of the Riesz operator) satisfy the following properties*

1. (translation invariance)  $F(\Omega) = F(\Omega + x)$  for all  $x \in \mathbb{R}^d$ .
2. (invariance under orthonormal transformations)  $F(\Omega) = F(T(\Omega))$  for every orthonormal transformation  $T$ .
3. (homothety law)  $F(k\Omega) = k^\alpha F(\Omega)$  for any  $k > 0$ .
4. (domain monotony) If  $A \subset B$  are open sets, then  $F(A) \leq F(B)$ .

**Proof:** The properties for  $\lambda_1$  are easily established starting from the variational formulation of  $\lambda_1$ . In the case of the Schatten norm it is more convenient to start from the expression (2.3.9). We sketch the proof of properties 1 and 3 for the functional  $\lambda_1(\cdot)$ . Let  $z \in \mathbb{R}^d$ , for this note that  $u \mapsto w$ , given by  $w(x) := u(x + z)$ , is an isomorphism from  $L^2(\Omega)$  to  $L^2(\Omega + z)$  and under this isomorphism, we have

$$\begin{aligned} \frac{\int_{\Omega+z} \int_{\Omega+z} \frac{w(x)w(y)}{|x-y|^{d-\alpha}} dx dy}{\int_{\Omega+z} |w(x)|^2 dx} &= \frac{\int_{\Omega+z} \int_{\Omega+z} \frac{u(x+z)u(y+z)}{|x-y|^{d-\alpha}} dx dy}{\int_{\Omega+z} |u(x+z)|^2 dx} \\ &= \frac{\int_{\Omega} \int_{\Omega} \frac{u(t)u(r)}{|t-r|^{d-\alpha}} dt dr}{\int_{\Omega} |u(t)|^2 dt} \end{aligned}$$

from which property 1 follows by taking the maximum. Now we show property 3, for this note that  $u \mapsto w$ , given by  $w(x) := u\left(\frac{x}{k}\right)$ , is an isomorphism from  $L^2(\Omega)$  to  $L^2(k\Omega)$  and under this isomorphism, we have

$$\begin{aligned} \frac{\int_{k\Omega} \int_{k\Omega} \frac{w(x)w(y)}{|x-y|^{d-\alpha}} dx dy}{\int_{k\Omega} |w|^2(x) dx} &= \frac{\int_{k\Omega} \int_{k\Omega} \frac{u\left(\frac{x}{k}\right)u\left(\frac{y}{k}\right)}{|x-y|^{d-\alpha}} dx dy}{\int_{k\Omega} \left|u\left(\frac{x}{k}\right)\right|^2 dx} \\ &= k^\alpha \frac{\int_{\Omega} \int_{\Omega} \frac{u(z)u(r)}{|z-r|^{d-\alpha}} dz dr}{\int_{\Omega} |u(z)|^2 dz} \end{aligned}$$

from which property 3 follows by taking the maximum. Properties 2 and 4 can be proved analogously. ■

## 2.4 Isoperimetric inequalities

In this section we recall some isoperimetric inequalities, both local and nonlocal, already known in the literature along with short sketches of their proofs. A few of the most basic isoperimetric inequalities are the following:

$P(A^*) \leq P(A)$  and  $P(\mathcal{S}A) \leq P(A)$  for any Borel measurable set in  $\mathbb{R}^d$  of finite perimeter.

Closely related to these isoperimetric inequalities are the Pólya-Szegő and Riesz inequalities.

**Theorem 2.4.1. (local Pólya-Szegő inequality)** *Let  $u$  be a non-negative function belonging to  $H_0^1(\mathbb{R}^n)$ . Then*

$$\int_{\mathbb{R}^n} |\nabla u^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx.$$

**Proof:** The following proof is from Lieb and Loss [40]. Let  $\hat{u}$  be the Fourier transform of  $u$ . Then, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx &= \int_{\mathbb{R}^n} |\widehat{\nabla} u(x)|^2 dx = \int_{\mathbb{R}^n} 4\pi^2 |\xi| |\hat{u}(\xi)|^2 d\xi \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_{\mathbb{R}^n} (1 - e^{-4\pi^2 |\xi|^2 t}) |\hat{u}(\xi)|^2 d\xi \\ &= \lim_{t \rightarrow 0} \left( \frac{1}{t} \int_{\mathbb{R}^n} |\hat{u}|^2 - \frac{1}{t} \int_{\mathbb{R}^n} \hat{u}(\xi) (G_t * u)(\xi) d\xi \right) \\ &= \lim_{t \rightarrow 0} \left( \frac{1}{t} \int_{\mathbb{R}^n} |u|^2 - \frac{1}{t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_t(x-y) u(x) u(y) dx dy \right) \\ &\geq \lim_{t \rightarrow 0} \left( \frac{1}{t} \int_{\mathbb{R}^n} |u^*|^2 - \frac{1}{t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u^*(x) G_t(x-y) u^*(y) dx dy \right) \end{aligned}$$

where  $G_t(x-y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}$  is the heat kernel. The inequality in the last line is a consequence of the Riesz inequality (2.1.12). ■

**Isoperimetric inequality for the fractional perimeter:** The following scale-invariant nonlocal isoperimetric inequality

$$\frac{P_s(E)}{|E|^{1-\frac{s}{d}}} \geq \frac{P_s(B)}{|B|^{1-\frac{s}{d}}} \quad (2.4.1)$$

where  $B$  is any  $d$ -dimensional ball, holds with equality occurring if only if  $E$  is a ball. This result follows directly from the fractional Sobolev inequality Theorem 4.1 proved in Frank and Seiringer [24] which they proved using a fractional Pólya-Szegő inequality on  $L^p$ . If we restrict ourselves to sets of finite perimeter in  $\mathbb{R}^d$ , then the above result also can be obtained following Proposition 4.2 and Corollary 4.4 of Brasco et al. [10]. The nonlocal isoperimetric inequality may be viewed as saying that the symmetric-decreasing rearrangement diminishes the fractional perimeter. As regards the Steiner symmetrization we may state the following and it, basically, follows directly from Corollary 4.1.2 (which is proved in Chapter 4 of this thesis) for the choice of the function  $\chi_E$  in  $W^{s,1}(\mathbb{R}^d)$ .

$$P_s(\mathcal{S}E) \leq P_s(E) \text{ for every Borel set } E \text{ of finite perimeter} \quad (2.4.2)$$

where  $\mathcal{S}E$  is Steiner symmetrization with respect a hyperplane.

**Theorem 2.4.2. (nonlocal Pólya-Szegő inequality under symmetric-decreasing rearrangement)** *Let  $d \geq 1$ ,  $0 < s < 1$ ,  $2s < d$  and  $u \in \widetilde{W}_0^{s,2}(\Omega)$ . Then,*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u^*(x) - u^*(y)|^2}{|x - y|^{d+2s}} dx dy \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx dy.$$

*The equality holds iff  $u$  is proportional to a translate of a symmetric-decreasing function.*

**Proof:** See [1] and [24] (where also the equality case is discussed). ■

**Remark 2.4.3.** *In [44] the nonlocal Pólya-Szegő inequality for Steiner symmetrization is stated without proof. A proof of this can be found in my master's thesis [45] (in Spanish). In Chapter 4, we give an English translation of the same for the sake of completeness.*

**(Rayleigh-Faber-Krahn inequalities):** We briefly recall the following Faber-Krahn inequalities.

**Theorem 2.4.4. (FKI for the Dirichlet Laplacian)** *Let  $\Omega$  be an open set of finite volume in  $\mathbb{R}^d$  and let*

$$\lambda_1(\Omega) = \inf \left\{ \int_{\Omega} |\nabla u(x)|^2 dx : u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1 \right\} \quad (2.4.3)$$

*be the first eigenvalue of the Laplacian eigenvalue problem on  $\Omega$  with Dirichlet boundary conditions*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.4.4)$$

*Then, we have*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

*where  $\Omega^*$  is the ball having the same volume as  $\Omega$ .*

**Proof:** Let  $\phi_1$  be a minimizer for  $\lambda_1(\Omega)$  in (2.4.3) so that

$$\lambda_1(\Omega) = \int_{\mathbb{R}^n} |\nabla \phi_1(x)|^2 dx. \quad (2.4.5)$$

Applying the Pólya-Szegő inequality (Theorem 2.4.1) together with the observation that the  $L^2$ -norm is unchanged under symmetric-decreasing rearrangement, that is, it holds that  $\|\phi_1\|_2 = \|\phi_1^*\|_2 = 1$ , we get

$$\lambda_1(\Omega) = \int_{\Omega} |\nabla \phi_1(x)|^2 dx \geq \int_{\Omega^*} |\nabla \phi_1^*(x)|^2 dx \geq \lambda_1(\Omega^*). \quad (2.4.6)$$

■

**Theorem 2.4.5. (FKI for the Riesz potential operator)** *Let  $\Omega$  be an open set of finite volume in  $\mathbb{R}^d$ . Abusing notation, let  $\lambda_1(\Omega)$  be the first eigenvalue of the Riesz potential operator given as in (1.2.2). Then, we have*

$$\lambda_1(\Omega^*) \geq \lambda_1(\Omega).$$

*Where  $\Omega^*$  is the symmetric rearrangement of  $\Omega$ .*

**Proof:** This is given following Lemma 3.2 of [51]. Let  $\phi_1$  be a maximizer of  $\lambda_1(\Omega)$ . Then, using Riesz's inequality for the symmetric-decreasing rearrangement (2.1.12) together with the observation that the  $L^2$ -norm is unchanged under symmetric-decreasing rearrangement, we have

$$\lambda_1(\Omega) = C(d, \alpha) \int_{\Omega} \int_{\Omega} \frac{\phi_1(x)\phi_1(y)}{|x-y|^{d-\alpha}} dx dy \leq C(d, \alpha) \int_{\Omega^*} \int_{\Omega^*} \frac{\phi_1^*(x)\phi_1^*(y)}{|x-y|^{d-\alpha}} dx dy \leq \lambda_1(\Omega^*). \quad (2.4.7)$$

■

**Theorem 2.4.6.** For  $\Omega \subset \mathbb{R}^d$  which is open and bounded, let  $\lambda_1^s(\Omega)$  be the principal Dirichlet eigenvalue for the fractional Laplacian defined as below

$$\lambda_1^s(\Omega) = \inf_{u \in \widetilde{W}_0^{s,2}(\Omega)} \left\{ [u]_{\widetilde{W}^{s,2}(\Omega)}^2 := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy : \|u\|_{L^2(\Omega)} = 1 \right\} \quad (2.4.8)$$

over the fractional order Sobolev space  $\widetilde{W}_0^{s,2}(\Omega)$ . Then, we have

$$\lambda_1^s(\Omega) \geq \lambda_1^s(\Omega^*). \quad (2.4.9)$$

Moreover, if equality holds in (2.4.9) then  $\Omega$  is a ball.

**Proof:** The inequality and the equality case, can be deduced using Theorem 2.4.2. ■

The following is an isoperimetric inequality for the Schatten norms of Riesz potential proved in Rozenblum et al. [51].

**Theorem 2.4.7.** For any integer  $p$  with  $p_0 := \frac{d}{\alpha} < p < \infty$ , we get

$$\|I_{\alpha,\Omega}\|_p \leq \|I_{\alpha,\Omega^*}\|_p.$$

**Proof:** By the trace formula for the Schatten norm of  $I_{\alpha,\Omega}$  given in Corollary 2.3.11, we get

$$\sum_{j=1}^{\infty} \lambda_j(\Omega)^p = \text{Tr}(I_{\alpha,\Omega}^p) = (C(d, \alpha))^p \int_{\Omega^p} \prod_{k=1}^p |x_k - x_{k+1}|^{\alpha-d} dx_1 \dots dx_p, \quad x_{p+1} \equiv x_1. \quad (2.4.10)$$

By Brascamp-Lieb-Luttinger inequality (2.1.13) for symmetric-decreasing rearrangement, we get

$$\begin{aligned} & \int_{\Omega} \dots \int_{\Omega} |y_1 - y_2|^{\alpha-d} |y_2 - y_3|^{\alpha-d} \dots |y_p - y_1|^{\alpha-d} dy_1 \dots dy_p \\ & \leq \int_{\Omega^*} \dots \int_{\Omega^*} |y_1 - y_2|^{\alpha-d} |y_2 - y_3|^{\alpha-d} \dots |y_p - y_1|^{\alpha-d} dy_1 \dots dy_p. \end{aligned}$$

By the above inequality and (2.4.10), we get

$$\sum_{j=1}^{\infty} \lambda_j^p(\Omega) \leq \sum_{j=1}^{\infty} \lambda_j^p(\Omega^*). \quad \blacksquare$$

## 2.5 Hausdorff convergence of sets

**Definition 2.5.1.** *The Minkowski addition of two sets  $X, Y \subset \mathbb{R}^d$  is defined by*

$$X \oplus Y := \bigcup_{y \in Y} (X + y). \quad (2.5.1)$$

*The Minkowski difference of two sets  $X, Y \subset \mathbb{R}^d$  is defined by*

$$X \ominus Y := \bigcap_{y \in Y} (X - y). \quad (2.5.2)$$

If  $Y = -Y$ , then

$$X \ominus Y = \bigcap_{y \in Y} (X + y). \quad (2.5.3)$$

**Note:** If  $K$  is a convex body the set  $K \ominus B(0, \epsilon)$  is called the inner parallel body of  $K$  at distance  $\epsilon$  (see pages 93 and 148 of [55]).

**Definition 2.5.2.** *Let  $K$  and  $C$  be two nonempty compact sets in  $\mathbb{R}^d$ . Then their Hausdorff distance is defined as*

$$d^H(K, C) = \inf\{\epsilon \geq 0; K \subseteq C \oplus B(0, \epsilon) \text{ and } C \subseteq K \oplus B(0, \epsilon)\}.$$

*Let  $O_1, O_2$  be two open subsets of a compact set  $B$ . Then the so-called complementary Hausdorff distance is defined by:*

$$d_H(O_1, O_2) = d^H(B \setminus O_1, B \setminus O_2). \quad (2.5.4)$$

**Remark 2.5.3.**  $d^H(K_n, K) \rightarrow 0$  when  $n \rightarrow \infty$  if and only if for every  $\epsilon > 0$  there is  $n_\epsilon$  such that  $K_n \subseteq K \oplus B(0, \epsilon)$  and  $K \subseteq K_n \oplus B(0, \epsilon)$  for every  $n \geq n_\epsilon$ .

The following compactness result with respect to Hausdorff convergence shall be useful.

**Theorem 2.5.4.** *Let  $B$  be a fixed compact set in  $\mathbb{R}^d$  and  $\{\Omega_n\}$  a sequence of open sets in  $B$ . Then, there is an open set  $\Omega \subset B$  and a subsequence  $\{\Omega_{n_k}\}$  of  $\{\Omega_n\}$  which converges with respect to the Hausdorff distance to  $\Omega$ .*

**Proof:** See Theorem 2.3.15 of Henrot [29]. ■

**Remark 2.5.5.** *Let  $\{P_k\}$  a sequence of polygons of  $n$  sides, contained in a closed ball  $B$ , then by Theorem 2.5.4 we have there is a set  $P$  such that a subsequence of  $\{P_k\}$  converges with respect to the Hausdorff distance to  $P$ . By the Heine-Borel theorem, up to a subsequence, the vertices of  $P_k$  converge to some number  $m$  of distinct points. So, in fact, these point should be the vertices of  $P$ , a polygon of  $m$  sides with  $m \leq n$ . In particular, it follows that if  $\{\Delta_n\}$  is a sequence of triangular domains with fixed area say  $A$  that converges with respect to the Hausdorff distance to a domain  $\Delta$ , then we have  $\lim_{n \rightarrow \infty} \text{Area}(\Delta_n) = \text{Area}(\Delta) = A$ . If  $\Delta$  is an interval, then we have  $\text{Area}(\Delta) = 0 < A$ . Therefore  $\Delta$  is a triangle. ■*

Finally, we end this section by stating a lower semicontinuity result for the perimeter with respect to the convergence, in the complementary Hausdorff distance, for which we refer to Henrot and Pierre [30, Section 2.3] for the details. Given a family of convex open sets  $\{\Omega_n\}$ , all contained in a fixed ball, that converges with respect to the complementary Hausdorff distance to an open convex set  $\Omega$  we have

$$P(\Omega) \leq \liminf_{n \rightarrow \infty} P(\Omega_n). \quad (2.5.5)$$



# Chapter 3

## Isoperimetric inequalities for the Riesz potential operator in the class of triangles and quadrilaterals

The main results of this chapter are the isoperimetric inequalities for the first eigenvalue and the Schatten norm of the Riesz potential operator while restricting the class of domains to be triangles or quadrilaterals. Some of the existing results have been reviewed briefly in the Introduction and recalled in Chapter 2.4. We recall that, given  $d > 1$ ,  $0 < \alpha < d$  and a bounded open domain  $\Omega \subseteq \mathbb{R}^d$ , the Riesz operator  $I_{\alpha,\Omega} : L^2(\Omega) \rightarrow L^2(\Omega)$  has been defined in Chapter 2.3 by

$$(I_{\alpha}u)(x) = C(d, \alpha) \int_{\Omega} \frac{u(y)}{|x - y|^{d-\alpha}} dy$$

where  $C(d, \alpha) = \pi^{-\frac{d}{2}} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}$ . This was seen to be a compact, self-adjoint, and positive operator whose maximal eigenvalue is given by

$$\lambda_1(\Omega) = \max \left\{ \int_{\Omega} \int_{\Omega} C(d, \alpha) \frac{u(x)u(y)}{|x - y|^{d-\alpha}} dx dy : u \in L^2(\Omega), \|u\|_{L^2(\Omega)} = 1 \right\}.$$

The maximizer in the above exists and is an eigenfunction which satisfies the following Euler-Lagrange equation

$$C(d, \alpha) \int_{\Omega} \frac{u(y)}{|x - y|^{d-\alpha}} dy = \lambda_1(\Omega)u(x) \text{ in } L^2(\Omega) \quad (3.0.1)$$

having the weak formulation

$$\int_{\Omega} \int_{\Omega} C(d, \alpha) \frac{u(x)\phi(y)}{|x - y|^{d-\alpha}} dx dy = \lambda_1(\Omega) \int_{\Omega} u(x)\phi(x) dx \text{ for all } \phi \in L^2(\Omega). \quad (3.0.2)$$

Moreover, the first eigenvalue is simple and the eigenfunction is of constant sign. The main aims of this chapter are to prove Theorems 1.2.1 and 1.2.2. We also prove two secondary results which are used in the proof of the main results. The first is the continuity of the functionals  $\lambda_1(\Omega)$  and  $\|I_{\alpha,\Omega}\|_p$  of the Riesz operator with respect to the convergence in the Hausdorff complementary metric of a family of uniformly bounded convex open sets (Proposition 3.1.4 and Corollary 4.2.3). The second is a discussion of the equality case in Riesz's inequality for the Steiner symmetrization in higher dimensions, a result which is not thoroughly treated in the literature.

### 3.1 Continuity of the shape functionals for the Riesz potential operator

Before we can prove the continuity results Proposition 3.1.4 and Corollary 4.2.3 for the spectral functionals for the Riesz operator we need the following observations on convex bodies.

The first proposition shows formally that the inner parallel body of a set  $X$  at distance  $\epsilon$  is contained in the complement, within any bigger set  $Z$ , of the fattening of  $Z \setminus X$  by  $\epsilon$ .

**Proposition 3.1.1.** *Let  $X, Z \subset \mathbb{R}^d$  with  $X \subset Z$  and let  $\epsilon > 0$ . Then*

$$X \ominus B(0, \epsilon) \subseteq Z \setminus ((Z \setminus X) \oplus B(0, \epsilon)). \quad (3.1.1)$$

**Proof:** From (2.5.2) it follows that  $(X \ominus B(0, \epsilon))^c = X^c \oplus B(0, \epsilon)$  from which

$$Z \cap (X \ominus B(0, \epsilon))^c = Z \cap (X^c \oplus B(0, \epsilon)). \quad (3.1.2)$$

On the other hand, we have trivially

$$Z \cap ((Z \cap X^c) \oplus B(0, \epsilon)) \subseteq Z \cap (X^c \oplus B(0, \epsilon)).$$

So, after taking the complement with respect to  $Z$  in the previous relation and then, using (3.1.2), we get

$$Z \setminus (Z \cap (X \ominus B(0, \epsilon))^c) \subseteq Z \setminus (Z \cap ((Z \cap X^c) \oplus B(0, \epsilon))).$$

From this, in view of the hypothesis that  $X \subseteq Z$ , we get

$$X \ominus B(0, \epsilon) \subseteq Z \setminus ((Z \cap X^c) \oplus B(0, \epsilon)). \quad \blacksquare$$

**Proposition 3.1.2.** *Let  $X$  be an open convex set of  $\mathbb{R}^d$ . Then, the following holds:*

$$X \ominus B(0, \epsilon) = \overline{X} \ominus B(0, \epsilon).$$

**Proof:** On the one hand, it is clear that  $X \ominus B(0, \epsilon) \subseteq \overline{X} \ominus B(0, \epsilon)$ .

On the other hand, for any  $x \in \overline{X} \ominus B(0, \epsilon)$ , it follows from the definition (2.5.2) that  $B(x, \epsilon) \subseteq \overline{X}$ . Since for an open convex set it is true that  $\overline{\dot{X}} = X$  (see Theorem 2.28 of [47]) we get  $B(x, \epsilon) \subseteq X$ , and so,  $x \in X \ominus B(0, \epsilon)$ . This proves the inclusion which is less obvious.  $\blacksquare$

The main ingredient in the proof of Proposition 3.1.4 is the following Lemma.

**Lemma 3.1.3.** *Let  $K$  be a convex body in  $\mathbb{R}^d$ , with  $B(0, r) \subset K \subset B(0, R)$  for some numbers  $r > 0$  and  $R > 0$ . If  $0 < \epsilon < \frac{r^2}{4R}$ , then*

$$\left(1 - 4\frac{R\epsilon}{r^2}\right) K \subset K \ominus B(0, \epsilon) \subset K. \quad (3.1.3)$$

**Proof:** See Lemma 2.3.6, page 93 of [55].  $\blacksquare$

The following abstract continuity property for a shape functional can be deduced, essentially following Proposition 1.3 [44] or Proposition 2.9 [42].

**Proposition 3.1.4.** *Let  $F$  be a functional over the classes of nonempty convex open subsets in  $\mathbb{R}^d$  which satisfies the following conditions*

1. *(invariance under translations)  $F(B+x) = F(B)$  for any  $x \in \mathbb{R}^d$  and  $B$  open convex subset of  $\mathbb{R}^n$ .*
2. *(domain monotonicity)  $F(B) \leq F(A)$  whenever  $A \subseteq B$  (or,  $F(A) \leq F(B)$  whenever  $A \subseteq B$ ).*
3. *(homothety law) There exist  $\alpha \in \mathbb{R}$ , such that  $F(kA) = k^\alpha F(A)$  for all  $k > 0$  and for every bounded open set  $A$ .*

*Let  $B$  be a fixed compact set in  $\mathbb{R}^d$  and  $\Omega_n$  be a family of nonempty convex open subsets of  $B$  which converges, for the complementary Hausdorff distance, to a nonempty convex open set  $\Omega$ . Then,  $F(\Omega) = \lim_{n \rightarrow \infty} F(\Omega_n)$ .*

**Proof:** Since  $F$  is invariant under translation we can assume that  $0 \in \Omega$ . Since  $\Omega$  is an open set, there is an open ball such that  $B(0, r) \subseteq \Omega$ . We assume, without loss of generality, that  $B$  is the closure of the ball  $B(0, R)$  for some  $R$  large enough. **STEP 1:** Since,  $d_H(\Omega_n, \Omega) \rightarrow 0$ , by the definition of the complementary Hausdorff distance, for any  $\epsilon > 0$  there exist  $n_\epsilon$  such that

$$B \setminus \Omega \subset (B \setminus \Omega_n) \oplus B(0, \epsilon) \text{ for all } n \geq n_\epsilon \quad (3.1.4)$$

and

$$B \setminus \Omega_n \subset (B \setminus \Omega) \oplus B(0, \epsilon) \text{ for all } n \geq n_\epsilon. \quad (3.1.5)$$

Further, by taking the relative complement in (3.1.5) with respect to  $B$  and thereafter applying Proposition 3.1.1 with the choices  $X = \Omega$  and  $Z = B$ , we obtain

$$\Omega \ominus B(0, \epsilon) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon. \quad (3.1.6)$$

Therefore, by Proposition 3.1.2, we also have

$$\overline{\Omega} \ominus B(0, \epsilon) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon \quad (3.1.7)$$

From the above, by choosing  $0 < \epsilon < r$ , we get

$$B(0, r - \epsilon) = B(0, r) \ominus B(0, \epsilon) \subseteq \Omega \ominus B(0, \epsilon) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon.$$

So, if  $0 < \epsilon < \frac{r}{2}$ , then we shall also have

$$B(0, r/2) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon. \quad (3.1.8)$$

**STEP 2:** Let us now fix  $0 < \epsilon < \frac{r^2}{16R}$ . For this choice, we also have  $0 < \epsilon < \frac{r^2}{4R}$  and so, applying Lemma 3.1.3 with the compact set  $\overline{\Omega}$  in mind, we get

$$\left(1 - 16 \frac{R\epsilon}{r^2}\right) \Omega \subset \left(1 - 16 \frac{R\epsilon}{r^2}\right) \overline{\Omega} \subset \overline{\Omega} \ominus B(0, \epsilon). \quad (3.1.9)$$

So, using (3.1.7), it follows that

$$\left(1 - 16 \frac{R\epsilon}{r^2}\right) \Omega \subset \Omega_n \text{ for all } n \geq n_\epsilon. \quad (3.1.10)$$

Then, by using the domain monotonicity and homothety properties of  $F$ , we can obtain the inequality

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)^\alpha F(\Omega) \leq F(\Omega_n).$$

After taking the liminf, as  $n \rightarrow \infty$ , we obtain

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)^\alpha F(\Omega) \leq \underline{\lim}_{n \rightarrow \infty} F(\Omega_n). \quad (3.1.11)$$

If we now take the limit as  $\epsilon \rightarrow 0$  in (3.1.11) we get

$$F(\Omega) \leq \underline{\lim}_{n \rightarrow \infty} F(\Omega_n). \quad (3.1.12)$$

STEP 3: Arguing similarly as in Step 1, but starting from (3.1.4), we can also obtain the inclusion

$$\Omega_n \ominus B(0, \epsilon) \subseteq \Omega \text{ for all } n \geq n_\epsilon.$$

In view of (4.2.2) and since we have chosen  $0 < \epsilon < \frac{r^2}{16R}$ , by applying Lemma 3.1.3 with the compact set  $\bar{\Omega}_n$  in mind we obtain

$$\left(1 - 16\frac{R\epsilon}{r^2}\right) \Omega_n \subset \left(1 - 16\frac{R\epsilon}{r^2}\right) \bar{\Omega}_n \subset \bar{\Omega}_n \ominus B(0, \epsilon).$$

So, continuing similarly as in Step 2, we conclude that

$$\overline{\lim}_{n \rightarrow \infty} F(\Omega_n) \leq F(\Omega). \quad (3.1.13)$$

The desired result follows from (3.1.12) and (3.1.13). ■

**Corollary 3.1.5.** *The first eigenvalue  $\lambda_1(\cdot)$  and the Schatten norm  $\|I_{\alpha,\cdot}\|_p$  of the Riesz potential operator satisfy the hypotheses of Proposition 3.1.4 and, so we have that  $\lambda_1$  and  $\|I_{\alpha,\cdot}\|_p$  are continuous with respect to the convergence, in the complementary Hausdorff distance, of a family of uniformly bounded nonempty convex open sets.*

## 3.2 Case of equality in Riesz's inequality for Steiner symmetrization

We require the following lemma.

**Lemma 3.2.1.** *Let  $A \subseteq \mathbb{R}$  be a Lebesgue measurable set with  $|A| > 0$ . If  $A = A + x$  for some  $x \in \mathbb{R} - \{0\}$ , then  $|A| = \infty$ .*

**Proof:** Since  $|A| > 0$ , necessarily there exists an  $n \in \mathbb{Z}$  for which  $B := A \cap [n, n + 1]$  has positive measure. Notice that  $B + x \subseteq A + x = A$  and then, using induction, we also obtain  $B + mx \subseteq A$  for every  $m \in \mathbb{Z}$ . Now, we assume, without loss of generality, that  $x > 0$  and then choose  $M \in \mathbb{N}$  such that  $Mx > 1$ . Then it follows that the intervals  $[sMx + n, sMx + n + 1]$  are disjoint for distinct  $s \in \mathbb{Z}$ . Since,  $B + sMx \subset [sMx + n, sMx + n + 1]$ , we obtain that the sets  $B + sMx$  are disjoint for distinct  $s \in \mathbb{Z}$ . Therefore, necessarily it follows that  $|A| = \infty$ , since  $A$  contains infinitely many disjoint copies of  $B$ . ■

**Proposition 3.2.2.** *In addition to the hypotheses of Corollary 2.1.8, suppose that  $g$  is symmetric with respect to the hyperplane  $H$  and strictly decreasing in the orthogonal direction (moving away) and that both  $f$  and  $h$  are non-zero measurable functions, then there is equality in (2.1.11) if and only if there exists  $w \in \mathbb{R}^d$  of the form  $(0, 0, \dots, 0, k)$  for some  $k \in \mathbb{R}$ , such that  $f(x) = \mathcal{S}f(x - w)$  and  $h(x) = \mathcal{S}h(x - w)$  for almost all  $x \in \mathbb{R}^d$ .*

**Proof:** The equality case in the  $d$ -dimensional case, for  $d \geq 2$ , is discussed below. The proof, unlike the proof of the equality case in Riesz's inequality under the symmetric-decreasing rearrangement or Schwarz symmetrization sketched in Theorem 3.9 of Lieb and Loss [40], does not require induction on the dimension. We exploit, directly, the one-dimensional result. Since we have chosen an orthogonal coordinate system wherein  $H$  is the plane  $\{(x', 0) : x' = (x_1, x_2, \dots, x_{d-1}) \in \mathbb{R}^{d-1}\}$ , the hypotheses on  $g$  gives us  $g(z', \cdot) = \mathcal{S}g(z', \cdot)$  and so the equality  $I(f, g, h) = I(\mathcal{S}f, g, \mathcal{S}h)$  written as

$$\begin{aligned} & \int_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} \int_{\mathbb{R} \times \mathbb{R}} f(x', x_d) g(x' - y', x_d - y_d) h(y', y_d) dx_d dy_d dx' dy' \\ &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} \int_{\mathbb{R} \times \mathbb{R}} \mathcal{S}f(x', x_d) g(x' - y', x_d - y_d) \mathcal{S}h(y', y_d) dx_d dy_d dx' dy', \end{aligned}$$

with  $(x', y') \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$ . For any fixed  $(x', y') \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$ , by definition,  $\mathcal{S}f(x', \cdot)$  and  $\mathcal{S}h(y', \cdot)$  are the one-dimensional symmetric-decreasing rearrangements of  $f(x', \cdot)$  and  $h(y', \cdot)$  respectively. So, for any  $x', y' \in \mathbb{R}^{d-1}$ , Riesz's inequality applied to the functions  $f(x', \cdot)$ ,  $g(x' - y', \cdot)$  and  $h(y', \cdot)$  viewed as functions of the final variable gives us

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}} f(x', x_d) g(x' - y', x_d - y_d) h(y', y_d) dx_d dy_d \\ & \leq \int_{\mathbb{R} \times \mathbb{R}} \mathcal{S}f(x', x_d) g(x' - y', x_d - y_d) \mathcal{S}h(y', y_d) dx_d dy_d. \end{aligned}$$

From the above it follows that

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}} f(x', x_d) g(x' - y', x_d - y_d) h(y', y_d) dx_d dy_d \\ &= \int_{\mathbb{R} \times \mathbb{R}} \mathcal{S}f(x', x_d) g(x' - y', x_d - y_d) \mathcal{S}h(y', y_d) dx_d dy_d \quad \text{a.e. } x', y'. \end{aligned} \quad (3.2.1)$$

Let  $S$  be the set of  $(x', y')$  for which equality holds in (3.2.1), so that  $(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}) \setminus S$  has measure 0. Also, let

$$M = \{x' \in \mathbb{R}^{d-1} : f(x', \cdot) \text{ is non-zero}\}, \quad N = \{y' \in \mathbb{R}^{d-1} : h(y', \cdot) \text{ is non-zero}\}.$$

Note that, by our hypothesis, both  $M$  and  $N$  are of positive measure. Also, for any  $(x', y') \in M \times N$ , by the definition of  $M$  and  $N$ , both  $f(x', \cdot)$  and  $h(y', \cdot)$  are non-zero functions. Therefore, for any  $(x', y') \in S \cap (M \times N)$ , since we have equality for Riesz's inequality in one dimension, we deduce that there exists  $k \in \mathbb{R}$  which, *a priori*, is a function  $k(x', y')$  of  $(x', y')$  such that

$$f(x', x_d) = \mathcal{S}f(x', x_d - k) \quad \text{a.e. } x_d \quad \text{and} \quad (3.2.2)$$

$$h(y', y_d) = \mathcal{S}h(y', y_d - k) \quad \text{a.e. } y_d. \quad (3.2.3)$$

Now,  $S \cap (M \times N)$  is of full measure in  $(M \times N)$  and so, for almost all  $y' \in N$  the section  $(S \cap (M \times N))_{y'}$  which is the section of  $S \cap (M \times N)$  at  $y'$  is of full measure in

$M$ . Consider any  $y'_0$  in  $N$  for which  $(S \cap (M \times N))_{y'_0}$  has the same measure as  $M$ . Now, for any  $x', z' \in (S \cap (M \times N))_{y'_0}$ , we observe that

$$\begin{aligned} f(x', x_d) &= \mathcal{S}f(x', x_d - k_{x', y'_0}) \quad a.e. \ x_d, \quad h(y'_0, y_d) = \mathcal{S}h(y'_0, y_d - k_{x', y'_0}) \quad a.e. \ y_d \\ f(z', x_d) &= \mathcal{S}f(z', x_d - k_{z', y'_0}) \quad a.e. \ x_d, \quad h(y'_0, y_d) = \mathcal{S}h(y'_0, y_d - k_{z', y'_0}) \quad a.e. \ y_d \end{aligned}$$

from which we obtain

$$h(y'_0, y_d) = h(y'_0, y_d + k_{z', y'_0} - k_{x', y'_0}) \quad a.e. \ y_d \quad (3.2.4)$$

Since  $y'_0$  belongs to  $N$  we see that  $\{y_d : h(y'_0, y_d) \neq 0\}$  is of positive measure. Also, since  $h(y'_0, \cdot)$  vanishes at infinity, it is possible to fix a  $t$  with  $0 < t < \infty$  such that the measure of  $A = \{y_d : h(y'_0, y_d) > t\}$  is finite and positive. Also notice that, by (3.2.4), we have  $A = A + k_{x', y'_0} - k_{z', y'_0}$ . Therefore, using Lemma 3.2.1, we have  $k_{z', y'_0} = k_{x', y'_0}$ . Let us denote this common value by  $k(y'_0)$ . This implies that  $f(x', \cdot)$  is symmetric-decreasing about  $x_d = k(y'_0)$  independently of  $x'$  in  $(S \cap (M \times N))_{y'_0}$ . That is,  $f(x', \cdot)$  is symmetric-decreasing about  $x_d = k(y'_0)$  for almost every  $x'$  in  $M$ . Since,  $f(x', \cdot)$  is independent of  $y'_0$  we can also conclude that  $k(y'_0)$  does not really depend on  $y'_0$  since a non-zero function vanishing at infinity cannot be symmetric-decreasing with respect to two distinct points. Therefore, for almost every  $x'$  in  $M$ ,  $f(x', \cdot)$  is symmetric-decreasing about  $x_d = k$  where  $k$  now is independent of  $x', y'$ .

Similarly, we could start with any  $x'_0 \in M$  such that the section  $(S \cap (M \times N))_{x'_0}$  which is the section of  $S \cap (M \times N)$  at  $x'_0$  is of full measure in  $N$ . Then, arguing as above, after reversing the roles of  $f$  and  $h$  and using the point  $x'_0$  in  $M$ , we also conclude that, for almost all  $w' \in M$ ,  $h(w', \cdot)$  is Steiner symmetric about  $y_d = k$  also, using (3.2.2)-(3.2.3), since  $k$  turns out to be independent of  $x'$  and  $y'$ . ■

### 3.3 Proof of the main theorems

The maximization problem of Theorem 1.2.1 for maximizing the Riesz eigenvalue  $\lambda_1(\cdot)$  in the class of triangles or quadrilaterals of given area may be formulated as

$$\max\{\lambda_1(P) : P \in P_N, \quad \text{area}(P) = a\} \quad (3.3.1)$$

where  $P_N$  is the class of nonempty open convex polygons with  $N$  edges. It is possible to show using the compactness of the class of uniformly bounded open convex sets for the complementary Hausdorff distance (see Theorem 2.3.15 of [29]) and Corollary 3.1.5 the existence of an open set  $\Omega$  for which the maximum is attained in (3.3.1). As mentioned in the introduction, even for the eigenvalues of the Laplacian with Dirichlet boundary condition the corresponding FKI is still open for polygons with 5 or more sides. Our aim is to show the existence of solutions of (3.3.1) in the class of triangles and quadrilaterals and to characterize them, that is, prove a FKI.

**Proof of Theorem 1.2.1** Let  $\Delta_1$  be an arbitrary triangle of area  $a$ . We successively define triangles  $\Delta_{n+1}$  by taking the Steiner symmetrization of  $\Delta_n$  with respect to the perpendicular bisector of a side with respect to which there is no symmetry. Then, by part 2 of the Proposition of 2.1.6, each of the triangles has area  $a$  and also the triangles are uniformly bounded since, by part 3 of the proposition by Proposition 2.1.14, the circumradius decreases after successive Steiner symmetrizations. Then, it can be shown

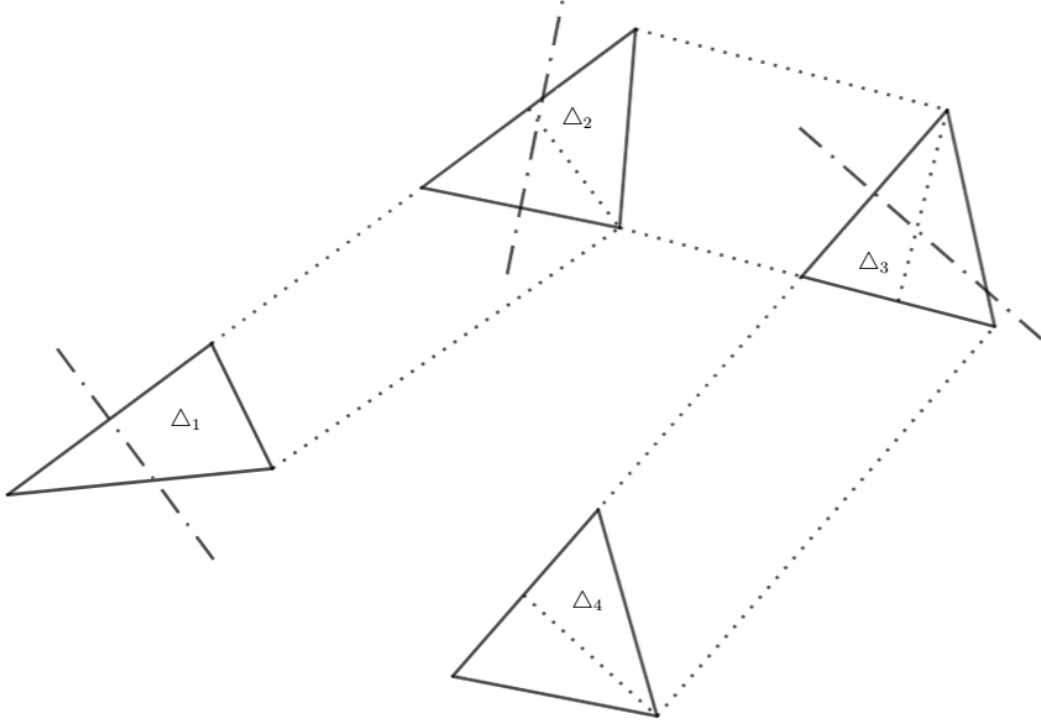


Figure 3.1: Sequence of triangles

that the sequence  $\Delta_n$  converges with respect to the complementary Hausdorff distance to an equilateral triangle  $\Delta$  (see page 158 of [46], 20.7 Theorem, pages 153-154 of [4] or Theorem 3.3.3 [29]).

Let  $f_n$  be an eigenfunction for  $\lambda_1(\Delta_n)$ , that is, a function for which the maximum is attained in (1.2.2) which we can take to be continuous, non-negative and having unit  $L^2$  norm. By property 1 of Proposition 2.3.13 the eigenvalue  $\lambda_1(\Delta_n)$  is invariant under translations, and so in this expression we may always assume that the coordinate system has its origin at the circumcenter of the triangle. Furthermore, for the ease of considering Steiner symmetrization of functions with respect to the chosen line of Steiner symmetrization of  $\Delta_n$ , by the rotation invariance of the eigenvalue, we may assume that this line is oriented along the  $x_1$ -axis. Let  $\mathcal{S}f_n$  be the corresponding Steiner symmetrization of  $\tilde{f}_n$  (the extension of  $f_n$  by zero outside  $\Delta_n$ ) which is a Borel measurable function vanishing at infinity. We note that  $\mathcal{S}f_n$  has to be supported on the closure of  $\Delta_{n+1}$ . By property 2 in Proposition 2.1.6, we note that  $\mathcal{S}f_n$  also has norm 1 in the  $L^2$  norm. Notice that the Riesz potential  $|x|^{-(2-\alpha)}$ , is Steiner symmetric with respect to the  $x_1$  axis. Since  $\alpha < 2$ , it is also a strictly decreasing function away from the  $x_1$ -axis in the  $x_2$ -direction. Therefore, we can apply Corollary 2.1.8 to the function  $\tilde{f}_n$  (taken twice) and with the function in the middle taken as the Riesz potential to obtain

$$\begin{aligned} \lambda_1(\Delta_n) &= \int_{\Delta_n} \int_{\Delta_n} C(2, \alpha) \frac{f_n(x)f_n(y)}{|x-y|^{2-\alpha}} dx dy \\ &\leq \int_{\Delta_{n+1}} \int_{\Delta_{n+1}} C(2, \alpha) \frac{\mathcal{S}f_n(x)\mathcal{S}f_n(y)}{|x-y|^{2-\alpha}} dx dy \end{aligned}$$

$$\begin{aligned}
&\leq \max_{w \in L^2(\Delta_{n+1})} \int_{\Delta_{n+1}} \int_{\Delta_{n+1}} C(2, \alpha) \frac{w(x)w(y)}{|x-y|^{2-\alpha}} dx dy \\
&= \lambda_1(\Delta_{n+1})
\end{aligned} \tag{3.3.2}$$

for each  $n$ . Therefore,

$$\lambda_1(\Delta_1) \leq \lambda_1(\Delta_n) \text{ for all } n.$$

We then use the continuity property of the Riesz eigenvalue for the complementary Hausdorff convergence (Corollary 4.2.3) and get

$$\lambda_1(\Delta_1) \leq \lim_{n \rightarrow \infty} \lambda_1(\Delta_n) = \lambda_1(\Delta).$$

This shows that  $\lambda_1$  has its maximum value for an equilateral triangle of given area.

The proof, in the case of quadrilaterals, that the maximum is attained for a square uses a similar argument as in the case of triangles. To start with, apply Steiner symmetrization with respect to an axis  $l_1$  which is perpendicular to a diagonal of the quadrilateral (not necessarily convex, see one of the figures below for the non-convex case), for which the other two vertices aren't on the same side of this diagonal. The resulting object is a *convex* quadrilateral which is symmetric with respect to this axis. Next, we Steiner symmetrize with respect to a perpendicular axis  $l_2$  and thereby get a rhombus. This is to be followed by a Steiner symmetrization with respect to an axis  $l_3$  perpendicular to one of the sides to produce a rectangle. The rectangle is then Steiner symmetrized with respect to an axis perpendicular to a diagonal to get, again, a rhombus. By repeating the procedures for the rhombus and rectangle we end up with an infinite sequence of rhombi and rectangles which converge, ultimately, in the complementary Hausdorff distance, to a square (refer to pages 158-159 of [46] or 20.8 Theorem, pages 154-155 of [4]).

Now we will address the proof of the uniqueness. We consider the case of triangular domains. Suppose that  $\Delta$  is any triangle for which the maximum is attained in (3.3.1). If we suppose that  $\Delta$  is not an equilateral triangle, then there is at least one side  $m$  of  $\Delta$  such that  $\Delta$  is not symmetric with respect to the perpendicular bisector to  $m$ . Let  $\mathcal{S}\Delta$  the Steiner symmetrization of  $\Delta$  respect to the perpendicular bisector of  $m$ . Let  $f$  be the first normalized eigenfunction associated to  $\lambda_1(\Delta)$  and  $\mathcal{S}f$  its Steiner symmetrization respect to the perpendicular bisector of  $m$ . Using the property 3 of the Proposition 2.1.6 and Corollary 2.1.8 and we obtain

$$\begin{aligned}
\lambda_1(\mathcal{S}\Delta) &\geq \int_{\mathcal{S}\Delta} \int_{\mathcal{S}\Delta} C(2, \alpha) \frac{\mathcal{S}f(x)\mathcal{S}f(y)}{|x-y|^{2-\alpha}} dx dy \\
&\geq \int_{\Delta} \int_{\Delta} C(2, \alpha) \frac{f(x)f(y)}{|x-y|^{2-\alpha}} dx dy \\
&= \lambda_1(\Delta).
\end{aligned} \tag{3.3.3}$$

Now, the fact that  $\Delta$  maximizes  $\lambda_1$  leads to the observation that  $\lambda_1(\mathcal{S}\Delta) = \lambda_1(\Delta)$  and so, we get the equality case in Riesz's inequality. Then, by Proposition 3.2.2 it follows that  $f$  is a translate of  $\mathcal{S}f$ . Furthermore,  $\mathcal{S}f$  is a maximizer for  $\lambda_1(\mathcal{S}\Delta)$ . As mentioned in the Proposition 2.3.12, the first eigenfunction for the Riesz operator  $f, \mathcal{S}f$  are strictly positive on  $\Delta$  and  $\mathcal{S}\Delta$  respectively, and so

$$\begin{aligned}
\Delta = \{x : f(x) > 0\} &= \{x : \mathcal{S}f(x-y) > 0\} \\
&= \mathcal{S}\Delta + y.
\end{aligned}$$



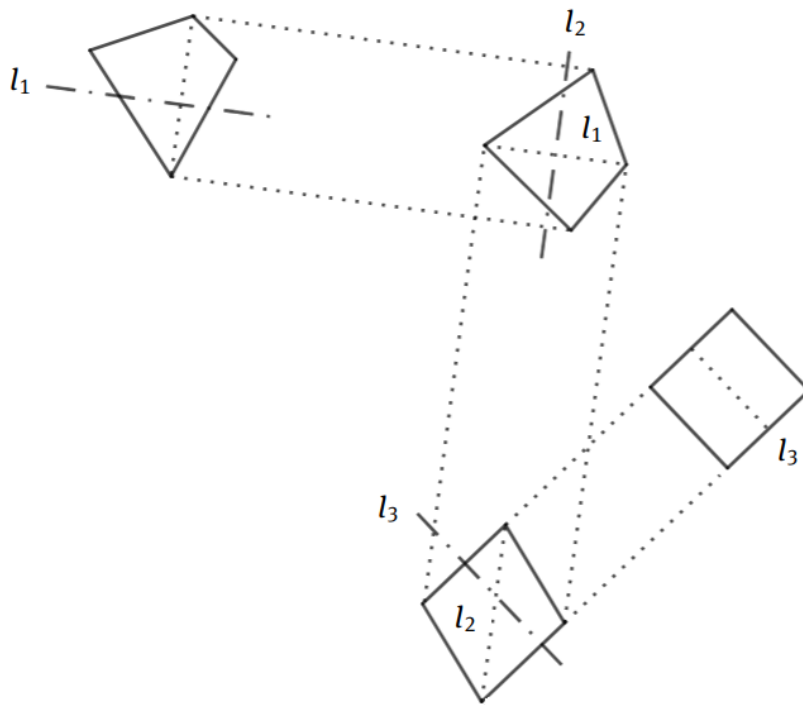


Figure 3.2: Sequence of quadrilaterals

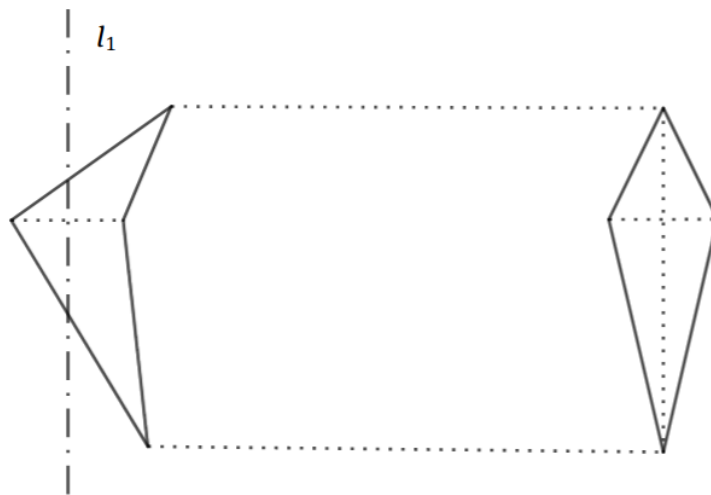


Figure 3.3: Symmetrization of a nonconvex quadrilateral

Thus,  $\Delta$  is Steiner symmetric with respect to the perpendicular bisector of  $m$  contrary to our supposition. So, we conclude that the equilateral triangle is the only minimizer. The uniqueness in the quadrilateral case is similarly proved by contradiction. ■

**Remark 3.3.1.** *Given an arbitrary pentagon, it is not possible to guarantee the existence of a line with respect to which the Steiner symmetrization results in a pentagon; in general, the number of sides will be increased. For example, the pentagon in the following figure*

becomes a hexagon. Note that the interval  $\overline{AB}$  gives two sides after applying a Steiner symmetrization with respect to  $l$ . This situation has been an obstacle in making progress on such isoperimetric problems for  $n > 4$ .

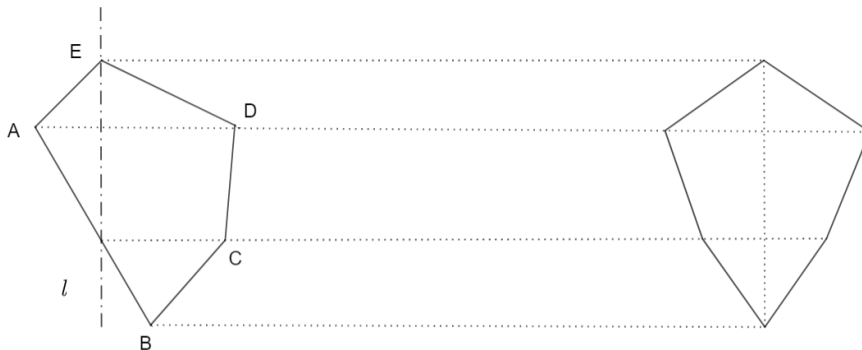


Figure 3.4: The Steiner symmetrization of a pentagon has, in general, at least six edges.

We now briefly consider possible 3–dimensional analogs of our planar results for triangles and quadrilaterals. These results apply to tetrahedra and certain prisms and are outlined in the following Theorem. We begin with tetrahedra, the 3–dimensional analogue of triangles. A reference to Steiner symmetrization in 3 dimension is p. 5 of [46].

**Definition 3.3.2.** Let  $E_1$  and  $E_2$  be two parallel planes (visualize the first one being under the second),  $R$  a polygonal region in  $E_1$ , and  $L$  a line that intersects  $E_1$  and  $E_2$ , but not  $R$ . For each point  $P$  of  $R$ , let  $\overline{PP'}$  be a segment parallel to  $L$  and joining the point  $P$  with its other extreme  $P'$  in  $E_2$ . The union of all segments  $\overline{PP'}$  is called a prism.

The polygonal region  $R$  is called the lower base, or simply the base of the prism. The part of the prism that is at  $E_2$  is called the top base. The edges of the faces of the prism that do not lie in the planes  $E_1$  and  $E_2$  are called the lateral edges of the prism. If  $L$  is perpendicular to  $E_1$  and  $E_2$ , then the prism is called a right prism; otherwise it is an oblique prism.

**Theorem 3.3.3.** The maximum of  $\lambda_1(\Omega)$  among all tetrahedra of given volume  $V$  is obtained when  $\Omega$  is a regular tetrahedron and only when  $\Omega$  is a regular tetrahedron. Similarly, the maximum of  $\lambda_1(\Omega)$  among all prisms (right or oblique) of given volume  $V$  and a quadrilateral base is obtained when  $\Omega$  is a cube and only when  $\Omega$  is a cube.

**Proof:** Let  $T$  be an arbitrary tetrahedron of a given volume  $V$ . First we Steiner symmetrize  $T$  with respect to a plane perpendicular to one of its edges. Next we Steiner symmetrize our new tetrahedron with respect to a plane perpendicular to the edge that lies in the former plane of symmetrization (this is the one edge of the tetrahedron that isn't the edge with which we started and that doesn't meet it). The tetrahedron so obtained has two planes of symmetry perpendicular to each other. By repeating these above two steps alternately we end up with an infinite sequence of tetrahedra which converge, ultimately, in the complementary Hausdorff distance, to a regular tetrahedron  $T_R$  with the given volume  $V$  (see [46], p. 159). By the same arguments given in the previous proof we have that

$$\lambda_1(T_R) \geq \lambda_1(T). \tag{3.3.4}$$

Now we will address the proof of the uniqueness of the case of tetrahedral domains. Suppose that  $T_0$  is any tetrahedron for which the maximum is attained in

$$\max\{\lambda_1(T) : T \text{ is a tetrahedron, } \text{volume}(P) = a\} \quad (3.3.5)$$

If we suppose that  $T_0$  is not a regular tetrahedron, then there is at least one edge  $m$  of  $T_0$  such that  $T_0$  is not symmetric with respect to the perpendicular bisector plane to  $m$ . Let  $\mathcal{S}T_0$  the Steiner symmetrization of  $T_0$  respect to the perpendicular bisector plane of  $m$ . Let  $f$  be the first normalized eigenfunction associated to  $\lambda_1(T_0)$  and  $\mathcal{S}f$  its Steiner symmetrization respect to the perpendicular bisector plane of  $m$ . Using property 3 of Proposition 2.1.6 and Corollary 2.1.8 and we obtain

$$\begin{aligned} \lambda_1(\mathcal{S}T_0) &\geq \int_{\mathcal{S}T_0} \int_{\mathcal{S}T_0} C(3, \alpha) \frac{\mathcal{S}f(x)\mathcal{S}f(y)}{|x-y|^{2-\alpha}} dx dy \\ &\geq \int_{T_0} \int_{T_0} C(3, \alpha) \frac{f(x)f(y)}{|x-y|^{2-\alpha}} dx dy \\ &= \lambda_1(T_0). \end{aligned} \quad (3.3.6)$$

Now, the fact that  $T_0$  maximizes  $\lambda_1$  leads to the observation that  $\lambda_1(\mathcal{S}T_0) = \lambda_1(T_0)$  and so, we get the equality case in Riesz's inequality. Then, by Proposition 3.2.2 it follows that  $f$  is a translate of  $\mathcal{S}f$ . Furthermore,  $\mathcal{S}f$  is a maximizer for  $\lambda_1(\mathcal{S}T_0)$ . As mentioned in the Proposition 2.3.12, the first eigenfunction for the Riesz operator  $f, \mathcal{S}f$  are strictly positive on  $T_0$  and  $\mathcal{S}T_0$  respectively, and so

$$\begin{aligned} T_0 = \{x : f(x) > 0\} &= \{x : \mathcal{S}f(x-y) > 0\} \\ &= \mathcal{S}T_0 + y. \end{aligned}$$

Thus,  $T_0$  is Steiner symmetric with respect to the perpendicular bisector plane of  $m$  contrary to our supposition. So, we conclude that the regular tetrahedron is the only maximizer.

The proof, in the case of prisms, uses a similar argument as in the case of tetrahedra. Let  $P$  be an arbitrary prism of a given volume  $V$ . First we Steiner symmetrize  $P$  with respect to a plane perpendicular to one of its lateral edges and change it into a right prism. We then take a sequence of Steiner symmetrizations with respect to planes orthogonal to the base of the right prism so that the bases converge to a square (it is enough to choose the same sequence of lines as in the successive Steiner symmetrizations of the quadrilateral and the perpendicular to the base to form these planes). This leads to a a right prism with square base (see [46], p. 159). The above process is repeated taking one of the lateral faces now as the base to get another right prism and so on until the right prisms converge to a cube.

The uniqueness in the case of prisms is proved similarly by contradiction. ■

**Proof of Theorem 1.2.2:** We now briefly sketch the proof of corresponding isoperimetric inequalities for Schatten norms of the Riesz potential. In all our discussions to follow,  $p$  is taken to be a positive integer. Other conditions will be included as needed. The proof goes along similar lines as that of Theorem 1.2.1, in the case of triangles we start with the same construction of a sequence of triangles  $\Delta_n$  starting from an arbitrary triangle. By the Brascamp-Lieb-Luttinger inequality for Steiner symmetrization (Theorem 2.1.9), we get

$$\sum_{j=1}^{\infty} \lambda_j^p(\Delta_n) = (C(2, \alpha))^p \int_{\Delta_n} \dots \int_{\Delta_n} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p$$

$$\leq (C(2, \alpha))^p \int_{\Delta_{n+1}} \dots \int_{\Delta_{n+1}} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p = \sum_{j=1}^{\infty} \lambda_j^p(\Delta_{n+1}).$$

We then use the continuity property of Schatten norms of the Riesz potential for the complementary Hausdorff convergence and get

$$\sum_{j=1}^{\infty} \lambda_j^p(\Delta_1) \leq \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} \lambda_j^p(\Delta_n) = \sum_{j=1}^{\infty} \lambda_j^p(\Delta).$$

The quadrilateral case is similarly proved.

Now we will address the proof of the uniqueness. We consider the case of triangular domains. Suppose that  $\Delta$  is any triangle for which the maximum is attained in  $\|I_{\alpha, \Omega}\|_p$  where  $\Omega$  has fixed area. If we suppose that  $\Delta$  is not an equilateral triangle, then there is at least one side  $m$  of  $\Delta$  such that  $\Delta$  is not symmetric with respect to the perpendicular bisector to  $m$ . Let  $\mathcal{S}\Delta$  the Steiner symmetrization of  $\Delta$  respect to the perpendicular bisector of  $m$ . We have,

$$\sum_{j=1}^{\infty} \lambda_j^p(\Delta) \leq \sum_{j=1}^{\infty} \lambda_j^p(\mathcal{S}\Delta).$$

Now, the fact that  $\Delta$  maximizes  $\|I_{\alpha, \Omega}\|_p$  leads to the observation that  $\|I_{\alpha, \Delta}\|_p = \|I_{\alpha, \mathcal{S}\Delta}\|_p$  and so, we get

$$\begin{aligned} & \int_{\Delta} \dots \int_{\Delta} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p \\ &= \int_{\mathcal{S}\Delta} \dots \int_{\mathcal{S}\Delta} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p \end{aligned} \quad (3.3.7)$$

On the other hand, for almost all  $y_{p-2}$  and  $y_1$ , there are no translates of  $|y_{p-1} - y_{p-2}|^{\alpha-2}$  and  $|y_p - y_1|^{\alpha-2}$  which are both Steiner symmetric. Therefore, by Theorem 3.2.2, we conclude that

$$\begin{aligned} & \int_{\Delta} \int_{\Delta} |y_{p-2} - y_{p-1}|^{\alpha-2} |y_{p-1} - y_p|^{\alpha-2} |y_p - y_1|^{\alpha-2} dy_{p-1} dy_p \\ & < \int_{\mathcal{S}\Delta} \int_{\mathcal{S}\Delta} |y_{p-2} - y_{p-1}|^{\alpha-2} |y_{p-1} - y_p|^{\alpha-2} |y_p - y_1|^{\alpha-2} dy_{p-1} dy_p, \end{aligned}$$

for almost every  $(y_1, y_2, \dots, y_{p-2})$ .

Integrating over the remaining variables we get

$$\begin{aligned} & \int_{\Delta} \dots \int_{\Delta} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p \\ & < \int_{\mathcal{S}\Delta} \dots \int_{\mathcal{S}\Delta} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p \end{aligned}$$

The above is contrary to (3.3.7). So, we conclude that the equilateral triangle is the only maximizer.

The uniqueness in the quadrilateral case is similarly proved by contradiction. ■

**Theorem 3.3.4.** *Let  $p$  be an integer such that  $p > \max(\frac{d}{\alpha}, 2)$ . The maximum of  $\|I_{\alpha, \Omega}\|_p$  among all tetrahedra of given volume is obtained when  $\Omega$  is a regular tetrahedron. Similarly, the maximum of  $\|I_{\alpha, \Omega}\|_p$  among all prisms (right or oblique) of given volume and a quadrilateral base is obtained when  $\Omega$  is a cube.*

**Proof:** The proof of this theorem is analogous to the proof of Theorem 3.3.3. ■

# Chapter 4

## An isoperimetric inequality for the fractional Cheeger constant and nonlocal perimeter for triangles and quadrilaterals

In this chapter we will prove Theorems 1.2.3 and 1.2.4. The proof uses a nonlocal Pólya-Szegő inequality under Steiner symmetrization and continuity results for the respective functionals, for the convergence of a family of uniformly bounded nonempty convex open sets, in the complementary Hausdorff distance. This inequality was also used in a previous work on isoperimetric inequalities for the first eigenvalue of the fractional  $p$ -Laplacian with Dirichlet boundary conditions [44]. Since, in the literature, it is difficult to find anything beyond a statement of the nonlocal Pólya-Szegő inequality under Steiner symmetrization its proof was detailed in the master's thesis of Olivares[44] in Spanish. Here, we reproduce the proof for the sake of completeness (only for bounded functions). The proof is an adaptation of the proof of Lemma A.2 of [24] to the case of Steiner symmetrization. The preliminary results include the continuity of the fractional perimeter and the continuity of the fractional Cheeger constant with respect to the convergence, in the complementary Hausdorff distance, of a family of uniformly bounded nonempty convex open sets.

### 4.1 Nonlocal Pólya-Szegő inequality

Following the same ideas given in Lemma A.2. of [24], for Steiner symmetrization instead of symmetric-decreasing rearrangement, we get the following lemma:

For  $J$  a nonnegative, convex function on  $\mathbb{R}^d$  with  $J(0) = 0$  and  $k$  a nonnegative measurable function on  $\mathbb{R}^d$ , we let

$$E[u] := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(u(x) - u(y))k(x - y)dx dy.$$

**Lemma 4.1.1.** *Let  $J$  be a nonnegative, convex function on  $\mathbb{R}$ ,  $J(0) = 0$  and  $k \in L_1(\mathbb{R}^d)$  be a nonnegative strictly decreasing function. Then for a bounded function  $u$  with  $E[u] < \infty$  and  $|\{u > \tau\}| < \infty$  finite for all  $\tau > 0$  we have*

$$E[u] \geq E[Su],$$

where  $Su$  is the Steiner symmetrization of  $u$  with respect a hyperplane  $H$ . If  $J(t) = |t|$ , then equality holds if and only if the level sets  $\{u > \tau\}$  are symmetric respect to  $H$  (up to a translation) for a.e.  $\tau > 0$ .

**Proof:** Let's write first  $J = J_+ + J_-$  with

$$J_+(t) = \begin{cases} J(t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

and

$$J_-(t) = \begin{cases} J(t) & \text{if } t < 0 \\ 0 & \text{if } t \geq 0. \end{cases}$$

We can define via the above

$$E_+[u] := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J_+(u(x) - u(y))k(x - y)dx dy. \quad (4.1.1)$$

Similarly, we define  $E_-$ . Following the above definition we can write  $E[u] = E_+[u] + E_-[u]$ . Then, it is enough to prove the theorem for the case  $E_+$ , because the case  $E_-$  follows similarly ( $k$  being symmetric respect to the center), which then will allow us to obtain the conclusion for  $E$ .

Since  $J$  is convex so is  $J_+$ . From the above  $J_+$  it is locally Lipschitz, so  $J_+$  is differentiable almost everywhere on  $[0, \infty)$  and  $J_+(t) = \int_0^t J'_+(\tau)d\tau$ , with  $J'_+$  the right derivative of  $J_+$ .

This allows us to write

$$\begin{aligned} \int_0^\infty J'_+(u(x) - \tau)\chi_{\{\tau:u(y)\leq\tau\}}(\tau)d\tau &= \int_{u(y)}^{u(x)} J'_+(u(x) - \tau)\chi_{\{\tau:u(y)\leq\tau\}}(\tau)d\tau \\ &= \int_{u(y)}^{u(x)} J'_+(u(x) - \tau)d\tau \\ &= - \int_{u(x)-u(y)}^0 J'_+(w)dw \\ &= J_+(u(x) - u(y)). \end{aligned} \quad (4.1.2)$$

Let

$$e_\tau^+[u] := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J'_+(u(x) - \tau)k(x - y)\chi_{\{\tau:u(y)\leq\tau\}}(\tau)dx dy.$$

Using Tonelli's Theorem, we get

$$\begin{aligned} \int_0^\infty e_\tau^+[u]d\tau &= \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J'_+(u(x) - \tau)k(x - y)\chi_{\{\tau:u(y)\leq\tau\}}(\tau)dx dy d\tau \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x - y) \int_0^\infty J'_+(u(x) - \tau)\chi_{\{\tau:u(y)\leq\tau\}}(\tau)d\tau dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J_+(u(x) - u(y))k(x - y)dx dy \\ &= E_+[u]. \end{aligned} \quad (4.1.3)$$

We claim that  $\int_{\mathbb{R}^d} J'_+(u(x) - \tau) dx < \infty$  for every  $\tau \in \mathbb{R}^+$ . Indeed, since  $u$  is bounded there is  $M \in \mathbb{R}^+$  such that  $u(x) \leq M$  for every  $x \in \mathbb{R}^d$ . From the above, together with the fact that  $J'_+$  is an increasing function and  $u$  is a function vanishing at infinity, we can get

$$\begin{aligned} \int_{\mathbb{R}^d} J'_+(u(x) - \tau) dx &= \int_{\{z:u(z) \geq \tau\}} J'_+(u(x) - \tau) dx \\ &\leq J'_+(M - \tau) \int_{\{z:u(z) \geq \tau\}} dx < \infty. \end{aligned}$$

Writing  $\chi_{\{u \leq \tau\}} = 1 - \chi_{\{u > \tau\}}$  and using that the Lebesgue integral is invariant under translation, we obtain that

$$\begin{aligned} e_\tau^+[u] &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J'_+(u(x) - \tau) k(x - y) \chi_{\{u \leq \tau\}}(y) dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J'_+(u(x) - \tau) k(x - y) (1 - \chi_{\{u > \tau\}})(y) dx dy \\ &= \|k\|_1 \int_{\mathbb{R}^d} J'_+(u(x) - \tau) dx - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J'_+(u(x) - \tau) k(x - y) \chi_{\{u > \tau\}} dx dy. \end{aligned} \quad (4.1.4)$$

By part 3 of Proposition 2.1.6 we have

$$\int_{\mathbb{R}^d} J'_+(u(x) - \tau) dx = \int_{\mathbb{R}^d} J'_+(\mathcal{S}u(x) - \tau) dx. \quad (4.1.5)$$

Applying Corollary 2.1.8 to the second term of (4.1.4) together with the relation (4.1.5) we get

$$e_\tau^+[\mathcal{S}u] \leq e_\tau^+[u]. \quad (4.1.6)$$

By (4.1.6) and the above we have that

$$E_+[\mathcal{S}u] \leq E^+[u],$$

now we will consider the case of equality. Now, let  $k$  be strictly decreasing and suppose that  $E_+[\mathcal{S}u] = E^+[u]$  for a bounded function  $u$ . It follows from the representation (4.1.3) that we have  $e_\tau^+[\mathcal{S}u] = e_\tau^+[u]$  for  $\tau > 0$  a.e. For equality to hold in (4.1.6), for any  $\tau > 0$  a.e. by Proposition 3.2.2 necessarily there exists an  $a_\tau \in \mathbb{R}^n$  such that

$$\chi_{\{u > \tau\}}(x', x_d) = \chi_{\{u > \tau\}}(x', x_d - a_\tau) \text{ and } J'_+(u(x', x_d) - \tau) = J'_+(\mathcal{S}u(x', x_d - a_\tau) - \tau).$$

for almost everywhere  $(x', x_d)$ . From this we conclude that  $\mathcal{S}\{u > \tau\} + (0, a_\tau) = \{u > \tau\}$  for a.e.  $\tau > 0$ . ■

The nonlocal Pólya-Szegő inequality (Theorem 2.4.2) can be rewritten by as follows

$$\int_{\mathbb{R}^d} (-\Delta)^s u^*(x) u^*(x) dx \leq \int_{\mathbb{R}^d} (-\Delta)^s u(x) u(x) dx \quad (4.1.7)$$

Recall the definition of fractional Laplacian and its relation to the Riesz potential operator

$$(-\Delta)^s u(x) = 2 \int_{\mathbb{R}^d} \frac{(u(x) - u(y))}{|x - y|^{n+2s}} dy := I_{-2s} u(x). \quad (4.1.8)$$

Using the above we can rewrite the inequality (4.1.9)

$$\int_{\mathbb{R}^d} I_{-2s} u^*(x) u^*(x) dx \leq \int_{\mathbb{R}^d} I_{-2s} u(x) u(x) dx \quad (4.1.9)$$

The following nonlocal Pólya-Szegő inequality under Steiner symmetrization is a consequence of Lemma 4.1.1. This inequality can be interpreted as the energy generated by the fractional  $p$ -Laplacian operator for  $p = 1$  (see [10]).

The proof of Proposition 4.1.2 is given in Theorem A.1 of [24] for the symmetric-decreasing rearrangement. We sketch the proof of the adaptation to the case of Steiner symmetrization given in [45] for the sake of completeness.

**Proposition 4.1.2.** *(nonlocal Pólya-Szegő inequality). Let  $d \geq 1$ ,  $0 < s < 1$  and  $u \in \widetilde{W}_0^{s,1}(\Omega)$  which is bounded. Then,*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\mathcal{S}u(x) - \mathcal{S}u(y)|}{|x - y|^{d+s}} dx dy \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|}{|x - y|^{d+s}} dx dy, \quad (4.1.10)$$

where  $\mathcal{S}u$  is the Steiner symmetrization of  $u$  with respect to a given hyperplane  $H$ . The equality holds iff the level sets  $\{u > \tau\}$  are symmetric with respect to  $H$  (up to a translation) for a.e.  $\tau > 0$ .

**Proof:** Since  $\mathcal{S}u(x)$  is nonnegative and  $||u(x)| - |u(y)|| \leq |u(x) - u(y)|$ , it suffices to prove the theorem for nonnegative functions. Using the definition of the Gamma function and a change of variables we obtain

$$\frac{1}{\Gamma(\frac{n+ps}{2})} \int_0^\infty \alpha^{\frac{n+ps}{2}-1} e^{-\alpha|x-y|^2} d\alpha = \frac{1}{|x-y|^{n+ps}}. \quad (4.1.11)$$

Using (4.1.11) and Tonelli's theorem for nonnegative integrands we have

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^{n+ps}} dx dy &= C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty \alpha^{\frac{n+ps}{2}-1} e^{-\alpha|x-y|^2} |u(x) - u(y)| d\alpha dx dy \\ &= C \int_0^\infty I_\alpha[u] \alpha^{\frac{n+ps}{2}-1} d\alpha, \end{aligned}$$

with

$$I_\alpha[u] := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)| e^{-\alpha|x-y|^2} dx dy \quad \text{and} \quad C = \frac{1}{\Gamma(\frac{n+ps}{2})}.$$

The function  $J(t) = |t|$  is convex and nonnegative with  $J(0) = 0$ . The function  $k(x) = e^{-\alpha|x|^2}$  is symmetric with respect to  $H$  (up to a translation) which belongs to  $L_1(\mathbb{R}^n)$ . Therefore, applying Lemma 4.1.1 to the functional  $I_\alpha$  we obtain the desired result.  $\blacksquare$

## 4.2 (Semi-)Continuity results for the shape functionals

We prove the continuity of the fractional perimeter with respect to the convergence in the Hausdorff complementary metric of a sequence of uniformly bounded convex open sets (for open sets in general we also prove a lower semicontinuity property) and the continuity of the fractional Cheeger constant with respect to the convergence in this metric.



**Proposition 4.2.1.** (lower semicontinuity of  $P_s$ ) Let  $B$  be a fixed compact set in  $\mathbb{R}^d$  and  $\Omega_n$  be a family of nonempty open subsets of  $B$  which converges, for the complementary Hausdorff distance, to a nonempty open set  $\Omega$ . Furthermore, if  $\Omega_n$  are sets of finite perimeter (classical perimeter) in  $\mathbb{R}^d$ , with

$$\sup_{n \in \mathbb{N}} P(\Omega_n) < \infty,$$

then

$$\underline{\lim}_{n \rightarrow \infty} P_s(\Omega_n) \geq P_s(\Omega). \quad (4.2.1)$$

**Proof:** By the compactness theorem for a family of sets with a uniformly bounded perimeter, there exists a subsequence  $\{\Omega_{n_k}\}$  such that  $\chi_{\Omega_{n_k}} \rightarrow \chi_\Omega$  in  $L^1(B)$  as  $k \rightarrow \infty$ . So, we can extract a further subsequence such that  $\chi_{\Omega_{n_k}} \rightarrow \chi_\Omega$  almost everywhere and, consequently,  $\chi_{\Omega_{n_k}^c} \rightarrow \chi_{\Omega^c}$  almost everywhere. By Fatou's lemma in the expression (2.2.1) for the fractional perimeter, we get the lower semicontinuity of  $P_s$

$$\begin{aligned} P_s(\Omega) &= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\chi_\Omega(x) \chi_{\Omega^c}(y)}{|x-y|^{d+s}} dx dy \\ &= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\underline{\lim}_{n_k \rightarrow \infty} (\chi_{\Omega_{n_k}}(x) \chi_{\Omega_{n_k}^c}(y))}{|x-y|^{d+s}} dx dy \\ &\leq \underline{\lim}_{n_k \rightarrow \infty} P_s(\Omega_{n_k}). \quad \blacksquare \end{aligned}$$

**Proposition 4.2.2.** Let  $B$  be a fixed compact set in  $\mathbb{R}^d$  and  $\Omega_n$  be a family of nonempty convex open subsets of  $B$  which converges, for the complementary Hausdorff distance, to a nonempty convex open set  $\Omega$ . Then,  $P_s(\Omega) = \lim_{n \rightarrow +\infty} P_s(\Omega_n)$ .

**Proof:** Arguing similarly as in Step 1 of the proof of Proposition 3.1.4 shows that there is an open ball  $B(0, r/2)$  such that

$$B(0, r/2) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon. \quad (4.2.2)$$

On the other hand,

$$P_s(\Omega_n) = 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\chi_{\Omega_n^c}(x) \chi_{\Omega_n}(y)}{|x-y|^{d+s}} dx dy. \quad (4.2.3)$$

It is not difficult to prove that  $\chi_{\Omega_n^c}(x) \chi_{\Omega_n}(y) \leq \chi_{B(0, r/2)^c}(x) \chi_B(y)$  and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\chi_B(x) \chi_{B(0, r/2)^c}(y)}{|x-y|^{d+s}} dx dy < \infty. \quad (4.2.4)$$

On the other hand, using the monotonicity of perimeters for convex bodies, we have  $P(\Omega_n) \leq P(\Omega)$  for all  $n \in \mathbb{N}$  (see [7], §7).

By the compactness theorem for a family of sets with a uniformly bounded perimeter (see Theorem 6.3 in Chapter 5 of [20]), there exists a subsequence  $\{\Omega_{n_k}\}$  such that  $\chi_{\Omega_{n_k}} \rightarrow \chi_\Omega$  in  $L^1(B)$  as  $k \rightarrow \infty$ . So, we can extract a further subsequence such that  $\chi_{\Omega_{n_k}} \rightarrow \chi_\Omega$  almost everywhere and, consequently,  $\chi_{\Omega_{n_k}^c} \rightarrow \chi_{\Omega^c}$  almost everywhere. Using the above and the dominated convergence theorem, we get  $P_s(\Omega) = \lim_{n \rightarrow +\infty} P_s(\Omega_n)$ .  $\blacksquare$

**Corollary 4.2.3.** *The functional  $h_s(\cdot)$  satisfies the hypotheses of Proposition 3.1.4, so we have that  $h_s$  is continuous with respect to the convergence, in the complementary Hausdorff distance, of a family of uniformly bounded nonempty convex open sets.*

**Theorem 4.2.4.** *Among all nonempty convex open sets in a ball  $B \subseteq \mathbb{R}^d$ , with the same nonlocal perimeter  $P_s$  then exists at least one with maximum volume.*

**Proof:** Let  $C$  be the class of all nonempty convex open sets in a ball  $B \subseteq \mathbb{R}^d$  with perimeter  $P_s$  equal to  $l$ . Now let  $\Omega_n$  be a maximizing sequence for  $V := \sup_{\Omega \in C} |\Omega| < \infty$ . Then, there exists a convex open  $\Omega \subset B$  and a subsequence in  $B$  that we name again  $\Omega_n$  such that  $d^H(\Omega_n, \Omega) \rightarrow 0$  as  $n \rightarrow \infty$  and  $\lim_{n \rightarrow \infty} |\Omega_n| = V$ . From Proposition 4.2.2 we get  $P_s(\Omega) = \lim_{n \rightarrow +\infty} P_s(\Omega_n) = l$ . From the continuity property of the volume with respect to the convergence, in the complementary Hausdorff distance, we have  $\lim_{n \rightarrow \infty} |\Omega_n| = |\Omega|$ . Therefore  $V = |\Omega|$ . ■

### 4.3 Proofs of the main theorems

**Proof of Theorem 1.2.3** Let  $\Delta_1$  be an arbitrary open triangle of area  $a$ . We, consider the sequence of open triangles  $\Delta_n$  defined by the successive Steiner symmetrizations in the proof of Theorem 1.2.1. By Proposition 2.2.4(3),  $P_s(\Delta_n)$  is invariant under translations, and so in this expression we may always assume that the coordinate system has its origin at the circumcenter of the triangle. Furthermore, for the ease of considering Steiner symmetrization of functions with respect to the chosen line of Steiner symmetrization of  $\Delta_n$ , by the rotation invariance of the  $P_s$ , we may assume that this line is oriented along the  $x_1$ -axis. Since  $\Delta_n$  has finite perimeter  $P(\Delta_n)$ , we can use the Proposition 2.2.3 to conclude that  $P_s(\Delta_n)$  is finite, so  $\chi_{\Delta_n} \in W_0^{s,1}(\Delta_n)$ . We can apply Proposition 4.1.2 to the function  $\chi_{\Delta_n}$ , obtaining

$$\begin{aligned} P_s(\Delta_n) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\chi_{\Delta_n}(x) - \chi_{\Delta_n}(y)|}{|x - y|^{2+s}} dx dy \\ &\geq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\chi_{S\Delta_n}(x) - \chi_{S\Delta_n}(y)|}{|x - y|^{2+s}} dx dy \\ &= P_s(\Delta_{n+1}) \end{aligned}$$

for each  $n$ . Therefore,

$$P_s(\Delta_1) \geq P_s(\Delta_n) \text{ for all } n.$$

Using Propositions 4.2.2 or 4.2.1, we have

$$P_s(\Delta_1) \geq \liminf_{n \rightarrow \infty} P_s(\Delta_n) \geq P_s(\Delta).$$

This shows that  $P_s$  attains its minimum value for an equilateral triangle. The proof, in the case of quadrilaterals, that the minimum is attained for a square uses a similar argument as in the case of triangles. The construction of the sequence of quadrilaterals is the same as in the proof of Theorem 1.2.1.

We now turn to the question of uniqueness in the case of triangular domains. Suppose that  $\Delta$  is an open triangle of given area for which the minimum is attained. If  $\Delta$  is not already an equilateral triangle, then there is at least one axis  $m$  (perpendicular to one

of its sides) such that  $\Delta$  is not Steiner symmetric with respect to  $m$ . Let  $\mathcal{S}\Delta$  be the Steiner symmetrization of  $\Delta$  respect to  $m$ . Without loss of generality,  $m$  passes through the origin.

$$\begin{aligned} P_s(\Delta) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\chi_\Delta(x) - \chi_\Delta(y)|}{|x - y|^{d+s}} dx dy \\ &\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\chi_{\mathcal{S}\Delta}(x) - \chi_{\mathcal{S}\Delta}(y)|}{|x - y|^{d+s}} dx dy \\ &= P_s(\mathcal{S}\Delta) \end{aligned}$$

Now, since we assumed that  $\Delta$  minimizes  $P_s$  this leads to  $P_s(\mathcal{S}\Delta) = P_s(\Delta)$  and so, we get the equality case in the nonlocal Pólya-Szegő inequality (Proposition 4.1.2). Then, by Proposition 4.1.2 it follows that

$$\mathcal{S}\{\chi_\Delta > \tau\} + a_\tau = \{\chi_\Delta > \tau\}, \quad (4.3.1)$$

for every  $\tau > 0$  a.e. Equation (4.3.1) is satisfied in particular for  $0 < \tau < 1$  a.e. Taking one of those  $\tau$ , let's call it  $\tau'$ , we have so

$$\mathcal{S}\Delta + a_{\tau'} = \Delta. \quad (4.3.2)$$

Thus,  $\Delta$  is Steiner symmetric with respect to  $m$ , contrary to our supposition. So, we conclude that the equilateral triangle is the only minimizer.

The quadrilateral case for  $P_s$  is proved similarly to the triangular case.  $\blacksquare$

**Theorem 4.3.1.** *The minimum of  $P_s(\Omega)$  among all tetrahedra of given volume is obtained when  $\Omega$  is a regular tetrahedron and only when  $\Omega$  is a regular tetrahedron. Similarly, the minimum of  $P_s(\Omega)$  among all prisms (right or oblique) of given volume and a quadrilateral base is obtained when  $\Omega$  is a cube and only when  $\Omega$  is a cube.*

**Proof:** The proof of this theorem is analogous to the proof of Theorem 3.3.3.  $\blacksquare$

**Proof of Theorem 1.2.4** Let  $E_{\Delta_n}$  be an  $s$ -Cheeger set of  $\Delta_n$ , that is,  $h_s(\Delta_n) = \frac{P_s(E_{\Delta_n})}{|E_{\Delta_n}|}$ . Then we have

$$\begin{aligned} h_s(\Delta_n) &= \frac{\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\chi_{E_{\Delta_n}}(x) - \chi_{E_{\Delta_n}}(y)|}{|x - y|^{2+s}} dx dy}{|E_{\Delta_n}|} \\ &\geq \frac{\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\chi_{\mathcal{S}E_{\Delta_n}}(x) - \chi_{\mathcal{S}E_{\Delta_n}}(y)|}{|x - y|^{2+s}} dx dy}{|\mathcal{S}E_{\Delta_n}|} \\ &= \frac{P_s(\mathcal{S}E_{\Delta_n})}{|\mathcal{S}E_{\Delta_n}|} \\ &\geq h_s(\mathcal{S}E_{\Delta_n}) = h_s(\Delta_{n+1}) \end{aligned} \quad (4.3.3)$$

for each  $n$ . Therefore,

$$h_s(\Delta_1) \geq h_s(\Delta_n) \text{ for all } n.$$

We then use the continuity property of  $h_s$  for the complementary Hausdorff convergence (Corollary 4.2.3) and get

$$h_s(\Delta_1) \geq \lim_{n \rightarrow \infty} h_s(\Delta_n) = h_s(\Delta). \quad (4.3.4)$$

This shows that  $h_s$  attains its minimum value for an equilateral triangle. The proof of the quadrilateral case is obtained in the same way as in the previous theorem.  $\blacksquare$

**Remark 4.3.2.** *The case of equality for the case of triangles and quadrilaterals the techniques given above, is not fully settled. Suppose that  $\Delta$  is an open triangle of given area for which the minimum is attained. If  $\Delta$  is not already an equilateral triangle, then there is at least one axis  $m$  (perpendicular to one of its sides) such that  $\Delta$  is not Steiner symmetric with respect to  $m$ . Let  $\mathcal{S}\Delta$  the Steiner symmetrization of  $\Delta$  respect to the perpendicular bisector of  $m$ . Let  $E_\Delta$  be  $s$ -Cheeger sets of  $\Delta$ . Following the same steps as in the proof of (4.3.3), we get*

$$\begin{aligned}
h_s(\Delta) &= \frac{P_s(E_\Delta)}{|E_\Delta|} = \frac{\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\chi_{E_\Delta}(x) - \chi_{E_\Delta}(y)|}{|x - y|^{2+s}} dx dy}{|E_\Delta|} \\
&\geq \frac{\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\chi_{\mathcal{S}E_\Delta}(x) - \chi_{\mathcal{S}E_\Delta}(y)|}{|x - y|^{2+s}} dx dy}{|\mathcal{S}E_\Delta|} \\
&= \frac{P_s(E_{\mathcal{S}\Delta})}{|E_{\mathcal{S}\Delta}|} \\
&\geq h_s(\mathcal{S}\Delta)
\end{aligned} \tag{4.3.5}$$

Now, the fact that  $\Delta$  minimizes  $h_s$  leads to the observation that  $h_s(\mathcal{S}\Delta) = h_s(\Delta)$  and so

$$\frac{P_s(E_\Delta)}{|E_\Delta|} = \frac{P_s(\mathcal{S}E_\Delta)}{|\mathcal{S}E_\Delta|}. \tag{4.3.6}$$

Considering  $|E_\Delta| = |\mathcal{S}E_\Delta|$ , and so, we get the equality case in the nonlocal Pólya-Szegő inequality. Therefore

$$E_\Delta = \mathcal{S}E_\Delta + a_{\tau'} \tag{4.3.7}$$

Though we cannot prove it, we suspect that  $\Delta = \mathcal{S}\Delta + a_{\tau'}$ .

**Theorem 4.3.3.** *The minimum of  $h_s(\Omega)$  among all tetrahedra of given volume is obtained when  $\Omega$  is a regular tetrahedron. Similarly, the minimum of  $h_s(\Omega)$  among all prisms (right or oblique) of given volume and a quadrilateral base is obtained when  $\Omega$  is a cube.*

**Proof:** The proof of this theorem is analogous to the proof of Theorem 3.3.3. ■

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